MASSACHUSETTS INSTITUTE OF TECHNOLOGY

GUIDANCE OF LOW-THRUST INTERPLANETARY VEHICLES

by

Edgar Dean Mitchell
Lieutenant Commander, USN

B.S., Carnegie Institute of Technology, 1952
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Doctor of Science

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Edgar D. Mitchell

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ABSTRACT

The problem studied in this thesis is the guidance of interplanetary vehicles which are thrusting for a large portion of the transfer. The vehicle is represented by a seven component state vector consisting of the position, velocity and mass of the spacecraft. The analysis is linearized by assuming that the actual state of the vehicle differs only a small amount from a known reference state. The reference trajectory is assumed to be a propellant-optimal path connecting the initial and final points.

The goal of the postulated guidance system is to satisfy position and velocity conditions at the target with minimum propellant expediture. Both fixed-time-of-arrival and variable-time-of-arrival guidance are discussed. Specification of the guidance criterion in the above manner permits the techniques of optimal control theory to be applied to the problem. Emphasis is placed on finding an analytic solution of the linearized equations. The desired solution is the control program which satisfies boundary conditions and minimizes propellant expenditure.

The method for solving the guidance problem is shown to be suitable as a technique for computing optimal reference trajectories. The trajectories are computed by iterative application of the guidance solution. Application of the guidance solution to the trajectory problem is shown to exploit an interpretation of the Euler equations which permits simplification of the computation technique.

The guidance solution is tested in a numerical example by using it to compute trajectories from Earth's orbit to the Martian orbit for different low-thrust vehicles.

The guidance solution is based on the assumption that vehicle state is known at the time a new control program is to be generated. Prior studies by several investigators detail methods of using celestial measurements to estimate state. A portion of this report is devoted to extending the method of celestial measurements to include measurement of engine performance. The additional measurement is shown to improve the estimate of state.

A discussion is presented of the difficulties arising from differences in the criterion for optimality as interpreted from the calculus of variations and from Pontryagin's maximum principle.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

A	seven by seven matrix of the partial derivatives of the state differential equations with respect to state variables
A*	six by six matrix of the partial derivatives of the equations of motion with respect to state variables
$\underline{a}$	three component thrust acceleration vector
B	seven by three matrix of the partial derivatives of the state differential equations with respect to control variables
C	three by three coordinate transformation matrix
$\underline{c}$	three component exhaust velocity vector
D	three by three transition matrix
$\underline{d}$	seven component vector of undetermined terminal constants
E	n by n covariance matrix of measurement error
F	n by k filter matrix
$\underline{f}$	three component thrust vector
G	three by three matrix of gravitational gradients
g_o	Earth gravity
H	six by six matrix of adjoint functions; scalar Hamiltonian
$\underline{i}$, i	unit vector in x direction; index number
I	n by n identity matrix
I_{sp}	specific impulse
J	scalar cost function
$\underline{j}$, j	unit vector in y direction; index number
$\underline{k}$, k	unit vector in z direction; index number
l	index number
M_o	seven by seven weighting matrix
M	six by six weighting matrix

M^*	six by six weighting matrix
MR	scalar mass ratio
m	normalized mass variable
N	n by k weighting matrix for measurement data
P	six by six skew symmetric matrix of identities
$\underline{p}$	n component adjoint vector
p	normalized power variable (scalar)
Q	six by six weighting matrix
$\underline{q}$	three component vector which is colinear with the control vector
$\delta\underline{q}$	n component measurement vector
R	nondimensional scalar parameter
$\underline{r}$	three component position vector
S	scalar cost function
$\underline{s}$	seven component state vector
T	seven by seven state transition matrix; scalar terminal time
t	time variable
U	n by n matrix of state uncertainties
$\underline{u}$	n component vector of state uncertainties; three component generalized control vector
$\underline{v}$	three component velocity vector
$\underline{x}$, x	n component generalized state vector; coordinate variable
y	coordinate variable
z	coordinate variable
$\underline{\alpha}$, α	six component vector function of time; specific mass of vehicle propulsion system
β	power plant mass fraction

$\underline{\gamma}, \gamma$	n component vector of scalar switching functions; switching function
Δ	increment of a quantity
δ	variation of a quantity
$\underline{\epsilon}, \epsilon$	n component vector of measurement error; measurement error
$\underline{\eta}$	six component miss vector which results if reference control is removed
θ	n by n diagonal matrix of correlation functions; heliocentric transfer angle
Λ	seven by seven matrix of adjoint functions
Λ^*	six by seven matrix of adjoint functions
Λ^{**}	six by six matrix of adjoint functions
$\underline{\lambda}$	n component vector of Euler variables
μ	gravitational constant for central body
$\underline{\nu}$	six component vector of Euler variable initial values
$\underline{\xi}$	six component miss vector which results if the reference control is used and a state variation occurs
$\underline{\pi}, \pi$	Lagrange multipliers
Σ	n by n matrix of measurement variance
σ	standard deviation of a measurement
τ	time variable
Φ	n by n fundamental matrix

The preceding symbols and the mathematical expressions used throughout the thesis will in general conform to the following rules:

	Example
1. A capital letter designates a matrix unless otherwise noted.	A

	Example
2. Underscored letters, both upper and lower case, represent column vectors.	$\underline{P}$, $\underline{r}$
3. Superscript T represents the transpose of a matrix or vector	A^T, $\underline{P}^T$
4. Superscript -1 represents the inverse of a matrix	A^{-1}
5. Unless underscored, lower case letters represent scalar quantities	r
6. Juxtaposition of matrix and vector symbols represents matrix multiplication	$A\,B$ $\underline{P}\,\underline{Q}^T$
7. The determinate of a matrix and the magnitude of a vector (when lower case letters are ambiguious) will be indicated by vertical bars.	$\lvert A \rvert$ or $\lvert \underline{r} \rvert$
8. Vertical brackets indicate a column vector composed of the enclosed quantities.	$\begin{Bmatrix} \underline{r} \\ \underline{v} \end{Bmatrix}$
9. Square brackets indicate a matrix whose elements are the quantities enclosed	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$
10. Diamond brackets indicate the time average of the enclosed quantities.	$< \epsilon \;\; \epsilon^T >$

The conventional dot notation is employed to indicate the time derivative of a quantity with respect to a non-rotating reference frame.

Subscripts are used to supplement the fundamental notation. Subscripted variables are defined as they are introduced.

CHAPTER I

INTRODUCTION

Early in the investigation of rocket propulsion for extra-terrestrial travel, it became apparent that conventional chemical propellants were inadequate for many interesting space missions [1,2,3,4]. This fact is due to the relatively low energy content per unit mass (specific energy) of chemical fuels. The theoretical mass ratios required for many interesting missions in the solar system approach numbers of the order of 10^3 and higher when chemical fuels are used[5].

The need for more efficient means of propulsion has led to the investigation of energy sources other than chemical reactions. Such studies have produced an entire spectrum of propulsive techniques[6], each having particular mission capabilities and each having its own theoretical and practical difficulties. In general, the use of higher specific energies is accompanied by a decrease in propellant flow rate and longer propulsion time for a given mass ratio. In limiting cases the propulsion time equals the transfer time. In addition, a longer propulsion time permits reduction of thrust levels for a given total impulse. The terms "continuous thrust" and "low thrust" stem from these two effects of high specific energy. Although the terms are not synonymous, they are often used interchangeably in the literature since they apply to the same types of vehicles. In the subsequent discussion, "continuous thrust" is used to describe vehicles which thrust for a major portion of the transfer and "low thrust" refers to vehicles which have acceleration levels less than about $10^{-3} g_o$.

This study is concerned primarily with those propulsion methods which rely upon a <u>separate energy source</u>* for the generation of propulsion energy. With few exceptions, these are low thrust devices.

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*Underlined words are defined in footnotes.

<u>separate energy source</u>: propulsion system characterized by a power plant which is independent of the thrust-producing mechanism.

Where it is necessary to be more specific, an on-board nuclear reactor is assumed.

We shall not be concerned with further justifying the use of these devices, nor with defining their regions of usefulness. Such questions are well covered in the literature[2,3,4,6]. The goal here is to assume that such vehicles will exist and to study the problem of guidance irrespective of the mission or of the particular propulsion method used.

1.1 The Guidance Problem

The presence of a thrust acceleration, acting over a significant portion of the spacecraft trajectory, introduces complexities in the analysis of spacecraft motion which make many of the techniques in common use for ballistic vehicles inapplicable. In particular, the elegant conic representations can be used only when the thrust may be treated as a small disturbing force. In general this is the case only in the region near a central body. Consequently it is desirable to formulate continuous-thrust investigations within a body of mathematics having sufficient applicability to treat most problems of interest. The calculus of variations and the concepts of optimal control theory meet this need.

In the terminology of this branch of applied mathematics, guidance of continuous-thrust spacecraft is the problem of finding a control program which transfers the spacecraft between two given points in space subject to constraints on the velocity at both terminal points. Since such programs are not unique, it is possible to place additional constraints on the solution such that some desired quantity (the cost function) is maximized or minimized. Thus, in addition to satisfying fixed boundary conditions at both launch and target points, solutions may be found which minimize transfer time, or propellant consumed or some

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guidance: as used in this thesis, the term refers to the process of determining a control program.

control: as used in this thesis, the term refers to the quantity or quantities (thrust or acceleration) representing the propulsive effort. It also refers to the mechanical process of directing the propulsive effort in accordance with the control program.

other quantity appropriate to the mission. This thesis is concerned with maximizing only the final mass. At the present time no method has been found which will prove rigorously that the solutions obtained in inverse square (and more general) gravitational fields satisfy the condition for an absolute maximum of the final mass[7]. One can only show that the solutions produce a maximum in the region of space under consideration, i.e. local maxima and then demonstrate from physical reasoning that other solutions are not likely.

From the viewpoint of guidance, initial conditions for the two-point boundary value problem are represented by the present state of the vehicle. If a vehicle can be flown in accordance with the solution that exists at the instant of launch, only one solution is needed. Such a case is a problem in control, not in guidance. That is, the desired thrust program is known (guidance) and the magnitude and direction of thrust must be controlled so that the spacecraft follows the desired trajectory (control). Even with "tight" control loops, however, perturbations inevitably occur which cause the spacecraft to depart from the original optimized trajectory. When this happens a new control program must be found which will cause transfer from the present state to the final state such that final mass is maximized. This is the problem we desire to solve. It follows, that if a solution can be found for the guidance problem, then by considering the launch state as the present state, the optimized trajectory connecting the launch point and target point may be determined. This latter application of the guidance techniques leads to that part of the thesis described as "trajectory computation."

In the preceding paragraph it was emphasized that the present state of the vehicle serves as the initial condition for the boundary value problem. It is necessary, therefore, that some method exist for determining the state of the vehicle at any time. In this thesis the unique features

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state: refers to the seven quantities which describe the vehicle in terms of physical variables

present state: refers to the vehicle state as determined by the space navigator, i.e. the state at the present instant of time. In this thesis navigation refers to the process of determining state from measurements.

of a continuous-thrust vehicle are examined from the viewpoint of estimating vehicle state.

Finally, the concept of constant exhaust power is examined as a criterion for low-thrust transfers. It is easily shown that if a linear system is power-limited, as in the case of separately powered rockets, the minimum expenditure of energy results from operation at maximum continuous power[7]. The application of this principle has been used extensively throughout the literature[8,9,10]. An analogous treatment of thrust-limited rockets is examined here.

1.2 Prior Studies

Published works on guidance of interplanetary low-thrust vehicles were almost nonexistent prior to early 1963. Miller[11] produced a guidance technique for cis-lunar space in his doctoral research in 1961. This technique consists of spiralling out from the Earth to some point from which the vehicle may coast to the vicinity of the moon, then matching the unique velocity vector, corresponding to the present position of the vehicle, which will result in achieving the target point. Miller showed that guidance of this type results in a relatively small fuel penalty for lunar missions.

Friedlander[10] formulated the problem in the classical calculus of variations and solved the adjoint equations in two dimensions for the sensitivity coefficients of the state variables along an optimized trajectory to Mars. In reference 12, he suggests a linearized solution which minimizes a quadratic function of the control variable variation. The solution approaches but does not attain the control program for maximum final mass. In reference 13 Friedlander applies his techniques to a vehicle using a Snap-8 power source.

The most recent work is that of Pfeiffer[14] who applies some of the newer developments in the theory of optimal control to the low-thrust guidance problem. Pfeiffer solves the guidance problem and minimizes

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power limited: refers to a propulsion system characterized primarily by a maximum power level.

thrust limited: refers to a propulsion system characterized primarily by a maximum thrust level

a penalty function which is "equivalent" to a quadratic form of the final state error. The control program produced by this method satisfies the boundary conditions "as closely as possible using the penalty function." (the quotation marks are from the reference) Pfeiffer's method, however, is not applicable to problems where certain boundary values are fixed.

Much of the recent work in optimal control theory concerns systems which have mathematical models similar to those for low-thrust vehicles. Many of the ideas developed in these investigations are directly applicable to the problem considered here. Significant contributions are attributable to Pontryagin[15], Kalman[16] and Breakwell[17] who have formulated existence theorems and derived necessary and sufficient conditions for optimal trajectories. Breakwell has also contributed important work in specifying the form of solutions with constrained control vectors. Athans, Falb and LaCross[18] working together and individually have solved many special cases of optimal trajectories for constrained control vectors.

Fundamental to any guidance study is a qualitative knowledge of the trajectory along which the space vehicle is to travel. Quite properly then, the earliest work in low-thrust propulsion consisted of studies of engine characteristics and trajectory characteristics. Several of the earlier studies of engine characteristics have been previously referenced. To those must be added the contributions of Langmuir[19] and Irving[20].

In the area of trajectory studies Tsien[21] performed some of the earliest work (1953). This was followed several years later with contributions by Lawden[22], Moeckel[23], Melbourne[24] and Zimmerman, McKay and Rossa[25]. The problem which confronted these authors was that analytic solutions to the trajectory problem can be found only for linear gravitational fields. For central force fields and more complex configurations, solving the two-point boundary value problem was a tedious trial-and-error procedure requiring the use of high speed digital computation. Satisfying an additional constraint for an optimized trajectory was tedious and time consuming even with high speed

computers. The work of Bryson[26] and others in the late 1950's and early 1960's served to simplify the machine procedures so that computation of optimal trajectories became less tedious and less time consuming.

1.3 Thesis Philosophy and the Method of Approach

Throughout the research and writing of this thesis the author has attempted to consider the low-thrust guidance problem from the viewpoint of a space navigator who is responsible for the safe and timely arrival of the spacecraft at the target point. The extrapolation of aircraft navigation experience into space navigation is, at best, a hazardous undertaking; however, it does provide a basis for certain decisions which have influenced the author's approach to this investigation. The following criteria were established from this philosophy and have been used when it became necessary to make definite assumptions.

1) The mission is manned, probably utilizing more than one vehicle, each of which is manned by several crewmen.

2) The mission duration is limited by consideration of human tolerances.

3) The spacecraft configuration and the mission have been specified. Hopefully, the spacecraft characteristics are optimum for the mission, but may not be.

4) Whatever the mission, at each of several points along the trajectory the navigator has three choices:
   a) to rendezvous with the target point at the preplanned time such that the trajectory minimizes propellant consumed.
   b) to rendezvous with the target point utilizing a time and trajectory such that propellant consumption is minimized.
   c) to rendezvous with the target point in minimum time utilizing the available propellant. This alternative is not treated in the thesis.

These guidelines establish the general context within which the guidance problem is to be solved. Let us now proceed to examine the specific items which complicate the solution.

1) The mathematical theory concerning optimal trajectories dictates that the first variation of the optimized quantity must vanish for the optimal path when the control is unconstrained[27,28,29]. As a consequence, if the optimized quantity is a state variable, the matrix of coefficients relating the control variables to the state variables is singular and its time integral along the optimal trajectory is also singular[12,14]. The singularity is proven in Appendix C. Solutions for the optimal control usually require inversion of this matrix. The problem of singular matrices is handled in this study by a method of deleting certain matrix elements which create the singularity and by the formation of a new matrix which can be inverted. The deletion method is an important part of the thesis and provides a general method for treating certain singularities without reformulating the problem.

2) Optimal trajectories for constrained control variables often possess discontinuities in the first derivatives of one or more state variables and in the control variables. This mathematical problem is handled by the use of switching functions which are continuous.

3) Optimal trajectories for constrained control variables require periods of maximum control magnitude. Therefore, if the maximum propulsive effort is required for the optimum trajectory, guidance around the optimum is limited to changes in thrust direction unless reserve propulsive power is available for guidance. The assumption that reserve power is available is used for this study.

It was stated in section 1.1 that the low-thrust guidance problem may be treated as a two-point boundary value problem in the calculus of variations[27,28,29]. Thus techniques of the calculus of variations constitute a primary mathematical tool. One of these techniques is the method of adjoints, which plays a fundamental part in subsequent chapters. The author has borrowed heavily from newer theories in optimal control[17,18] since the state space formulations widely used in the

literature of that field are applicable to low-thrust guidance. One of the more useful tools in optimal control theory is associated with Pontryagin[15] although other authors have used the same principle[16].

A derivation of Pontryagin's maximum principle is outlined in Appendix D for convenience of the reader.

To facilitate notation and to preclude the possibility of fundamental notions becoming obscured by the quantity of algebraic detail, matrix notation and the ideas of matrix calculus are used throughout. Finally, to test the thesis, numerical analysis and an IBM 7094 were employed.

1.4 Relationship to Prior Studies

In the research preceding this study, the author began with the formulations of Friedlander[10] and Melbourne[8,24], and attempted to extend their ideas into areas of more general application and to find solutions to the guidance problem which were useful from the space navigator's viewpoint. One of the fundamental considerations was to find control laws which optimize propellant consumption. Cost functions which produce near-optimal propellant consumption were rejected.

The idealized formulations for the separately powered rocket require a wide range of thrust and of specific impulse as the vehicle traverses its trajectory. Current technology indicates that variable-specific-impulse thrusters will not be available in the forseeable future, at least for electrostatic vehicles. To satisfy this engineering restriction, investigators have continued to use the power-limited formulations but approximate the optimal thrust magnitude programs with regions of constant specific impulse where relatively low values of specific impulse are required and with coast elsewhere. It is shown in this thesis that the so-called "bang-bang" control used to satisfy the engineering restriction can be derived by abandoning the concept of power-limited thrusting.

In section 1.1 the necessity of estimating the state is discussed. The works of Battin[30], Stern[31], and Potter and Stern[32] serve as the

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electrostatic (propulsion) : A propulsion method depending upon the acceleration of charged particles through an electrostatic field.

starting point for extending, to the low-thrust case, the navigation techniques (estimate of state) which have been developed for ballistic vehicles.

Finally, using the guidance solutions an iterative technique for computing low-thrust trajectories is developed. Recent investigations in the field of trajectory computation appear to rely heavily on steepest ascent techniques. These techniques require rather complex programs, careful handling and suffer from the problem of slow convergence as the optimal trajectory is approached. The technique suggested in this study appears to be faster even for discontinuous control variables.

1.5 Summary of Thesis Objectives

In closing this introductory chapter it seems appropriate to provide a sketch of the objectives of subsequent discussions. Briefly, the objectives of this thesis are: 1) to present a linear, noniterative method for deriving a control program for guidance of low-thrust space vehicles with respect to a propellant-optimal reference, 2) to show that the computed control programs satisfy the necessary conditions for an optimum, 3) to show that the method may be extended to trajectory computation by successive iteration of the guidance solution; 4) to examine a method of estimating the state of continuous-thrust vehicles and 5) to examine the concepts of power-limited and thrust-limited vehicles from the viewpoint of guidance.

The fundamental argument of the thesis may be extracted from these several objectives. It is: "There exists a linear method which produces a propellant-optimal control program in a noniterative form for guidance of power-limited and thrust-limited space vehicles, and which provides a simple, rapidly converging iterative technique for computing propellant-optimal trajectories."

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propellant-optimal: refers to a trajectory which results in minimum propellant usage.

CHAPTER II

THE LOW-THRUST SPACE VEHICLE

2.1 Summary of Chapter II

In this chapter the parameters which characterize low-thrust vehicle performance are derived and discussed. The constant-exhaust-power concept, which has been widely used in the past, is shown to be the optimum method of engine control. The propellant cost accruing from engineering restrictions on variable specific impulse is computed for field-free space by analytic methods. Both methods of engine control, the ideal and the practical, are discussed from the viewpoint of guidance. Finally, the results of a numerical study, which confirm the derivations, are presented.

2.2 The Use of a Separate Energy Source

One may verify from momentum considerations that the instantaneous force exerted on a space vehicle by its exhaust stream is:

$$\underline{f} = \dot{m}\,\underline{c} \qquad (2\text{-}1)$$

where $\underline{f}$ is the force vector, $\underline{c}$ is the oppositely directed exhaust velocity and $\dot{m}$ is the mass rate of the vehicle. ($-\dot{m}$ is the propellant flow rate.)

Since, the force magnitude may be held constant if $\dot{m}$ and $\underline{c}$ are varied inversely, the selection of a high exhaust velocity and low propellant flow tends to reduce the total amount of propellant required for a given impulse. Unfortunately, processes which use the products of combustion as the propellant are unable to produce, simultaneously, the low flow rates and high exhaust velocities required for many interesting missions. If we use specific impulse, I_{sp}, as a measure of the engine performance, where

$$I_{sp} = \frac{c}{g_o} = \frac{f}{(-\dot{m})\,g_o} \qquad (2\text{-}2)$$

and g_o is the acceleration due to gravity at the Earth's surface, then chemical systems are limited to values of specific impulse under 600 seconds and direct nuclear systems to values of about 1600 seconds[5]. However, if a separate energy source is available, the source energy may be converted into electrical energy and used to accelerate charged propellant particles. Values of specific impulse in the range of 3000 to 20,000 seconds appear attainable in this way[6].

Several energy sources and conversion processes are under investigation. These may be divided into the two broad categories of direct or indirect energy conversion. Direct methods may be characterized by the absence of a mechanical phase in the conversion process. Power from solar cells is one example of this type. Direct production of electrical energy in a nuclear reactor is another. A proposal for this latter method is discussed in reference 33 however it has not been proven in the laboratory.

Indirect conversion of the source energy into electrical energy appears, at the present time, to be more realistic for large manned spacecraft which have power requirements in the range of several megawatts. The process generally considered most promising uses a nuclear source to power turbomachinery for the production of electrical energy. A block diagram of this type system is presented in Figure 2-a.

In this report, an on-board energy source is explicitly assumed although much of the guidance analysis is independent of such considerations. This assumption permits the power availability to be independent of the trajectory. This is not the case, for example, when solar energy collectors are used since for a given collector area the power availability varies inversely with the square of the distance to the sun.

2.3 The Constant-Power Concept

Separate energy sources are most often described as power-limited devices. That is, their rate of energy conversion is the design criterion. If the energy produced by the source is all converted to propulsive energy then the power in the exhaust stream is given by

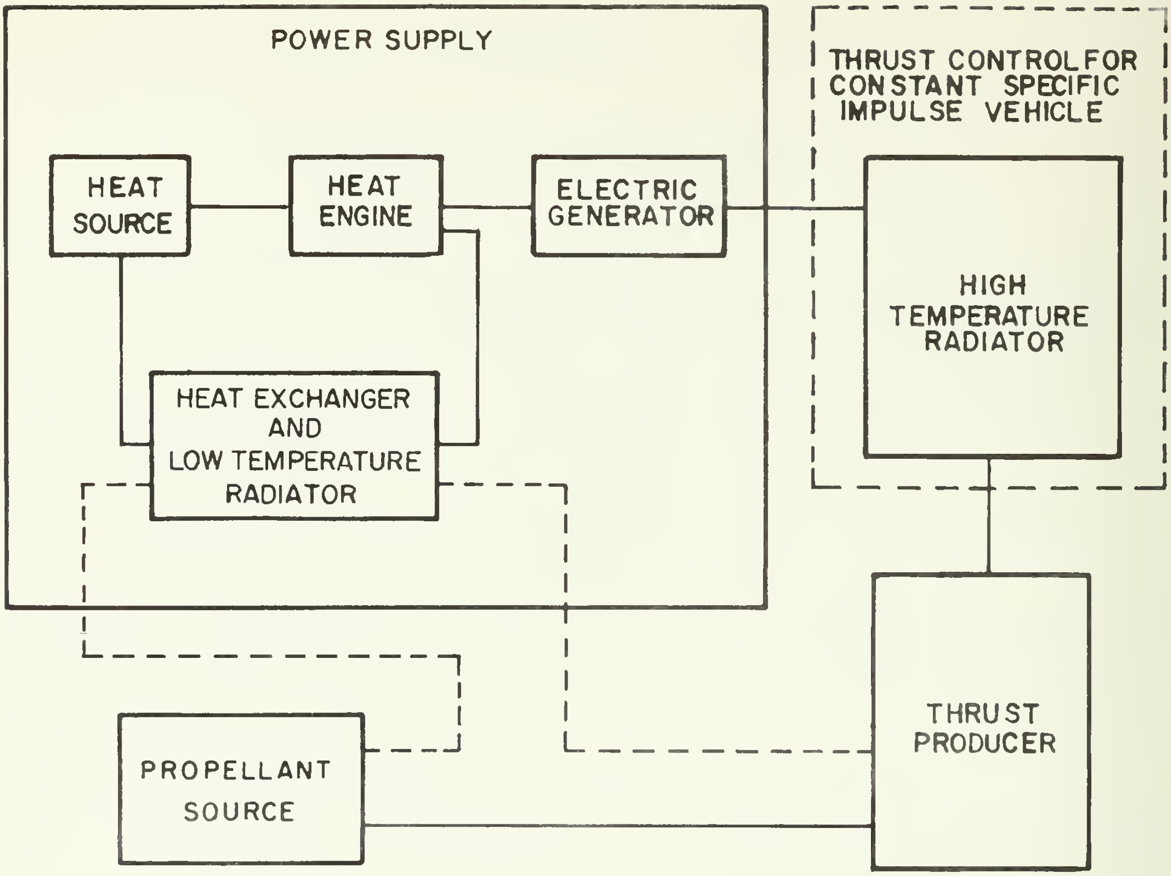

Fig. 2-a. Schematic diagram of separately powered rocket.

$$p = \frac{-\dot{m}\, c^2}{2} \tag{2-3}$$

(The efficiency of the conversion process will be introduced in a subsequent discussion.)

It is well known that for a large class of problems in linear fields, minimum total energy expenditure results from operation at maximum continuous power. This result may be obtained for conservative fields in general by use of variational techniques as shown in reference 7. The usual assumption in low-thrust investigations, that exhaust power is a constant and equal to its maximum value, is therefore, quite valid in the idealized problem. This result is used to establish parameters for the separately powered rocket.

Combining equation (2-3) with the relations

$$f = -\dot{m}\, c \tag{2-4}$$

$$f = m\, a \tag{2-5}$$

one obtains

$$\frac{-\dot{m}}{m^2} = \frac{a^2}{2p} \tag{2-6}$$

where a is the thrust acceleration of the vehicle. Integration of equation (2-6) over the flight path yields

$$\frac{1}{m_f} - \frac{1}{m_o} = \frac{1}{2p} \int_0^{t_f} a^2\, dt \tag{2-7}$$

where the subscripts indicate initial and final times. Equation (2-7) is more conveniently expressed in terms of the mass ratio.

$$MR - 1 = \frac{m_o}{2p} \int_0^{t_f} a^2\, dt \tag{2-8}$$

Three parameters will now be defined

$$J = \frac{1}{2} \int_0^{t_f} a^2\, dt \tag{2-9}$$

$$\alpha = \frac{m_E}{p} \tag{2-10}$$

$$\beta = \frac{m_E}{m_o} \tag{2-11}$$

J is the well known acceleration integral, α is the specific mass of the propulsion system and must include the total efficiency of energy conversion when p is the desired exhaust power; β is the propulsive system mass fraction and m_E is the propulsive system mass. Substituting these parameters into equation (2-8) one obtains

$$MR = 1 + \frac{\alpha}{\beta} J \tag{2-12}$$

For a spacecraft with fixed parameters and a specified mission, the minimum mass ratio and thus the minimum propellant usage is achieved for trajectories which make J a minimum.

The objective of trajectory computation and of guidance is to find a path which accomplishes the mission and minimizes J.

2.4 Limitation of the Constant-Power Concept

Using the equations of section 2.3, the thrust acceleration may be expressed as

$$a = \frac{2p}{cm} \tag{2-13}$$

Figure 2-b shows a plot of the thrust acceleration magnitude during a fast transfer to Mars for which J is a minimum. The absolute value

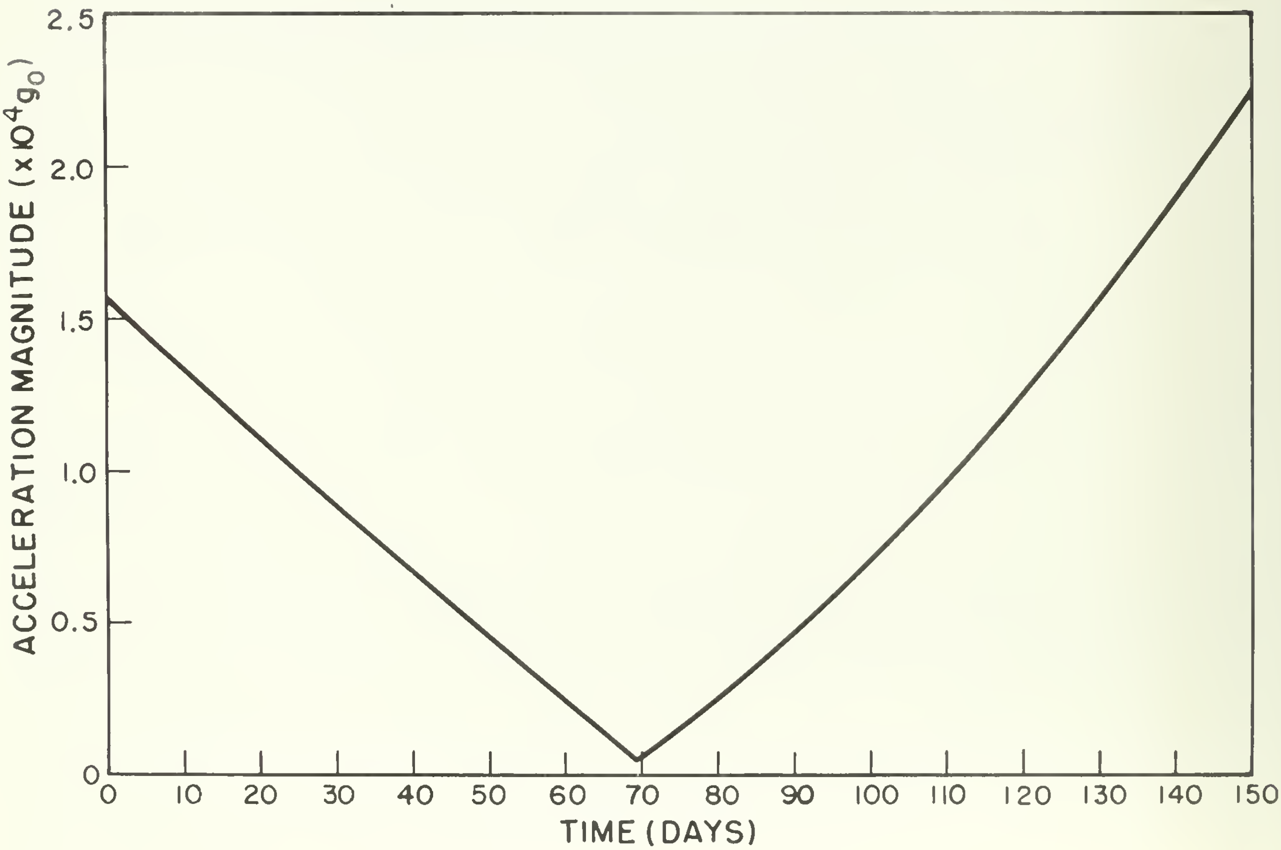

Fig. 2-b. Optimum acceleration level for variable-specific-impulse Earth-Mars transfer.

of acceleration has a variation over two orders of magnitude. From equation (2-13) it is apparent that the exhaust velocity, (i. e., the specific impulse) must have a similar range of values since m is monotonically decreasing. However, the current thinking of investigators who are studying propulsive devices indicates that in a given thrust producer, only a very small range of exhaust velocity variation will be feasible unless major advances in technology are forthcoming. The thrust producer can be designed for one particular exhaust velocity. Operation at other than the design point tends to decrease efficiency and shorten, markedly, the useful life of the device.

This restriction on exhaust velocity is treated in the literature[19] as a departure from the ideal conditions of constant exhaust power. The author treats this restriction from a slightly different viewpoint. In the succeeding section parametric equations for a constant-specific-impulse rocket will be derived which are analogous to equations (2-8) through (2-12). With these parametric equations one may compare directly the performance of a rocket with specified maximum power, under the two modes of propulsive control, i. e. constant-specific-impulse (CSI) or variable-specific-impulse (VSI).

2.5 The Constant-Specific-Impulse Concept

The propellant-optimal thrust program for a CSI rocket is shown subsequently to have two magnitudes, maximum thrust or zero thrust. In order to allow for maneuvering and for departures from the reference trajectory, reserve thrust capability is required. (For VSI rockets this problem does not occur since thrust is continuously variable.) A method for controlling thrust magnitude which appears quite attractive and is compatable with thruster design is to build up the propulsion unit from a large number of small thruster nozzles of constant specific impulse. Proposals of this type appear in the literature but are not analyzed in detail[35]. Control of thrust magnitude is obtained by controlling the number of thrusters in operation.

If a nuclear energy source is used the power range is not continuously variable from zero to maximum power and may be limited in the number of times that stopping and starting is permitted. However, the

energy source and the thruster units can be operated at their respective optimums by placing an external high temperature resistor in the system which will radiate energy in excess of that required by the thrusters. Since the nuclear fuel constitutes only a very small fraction of vehicle weight, this procedure has negligible effect on the final mass of the vehicle. Figure 2-a shows the propulsive system schematic diagram.

For a propulsion unit constructed in this manner, the total thrust of the vehicle is some fraction of the total thrust available. That is

$$f = -n\dot{m}_o c \qquad (2\text{-}14)$$

$$f_o = -\dot{m}_o c \qquad (2\text{-}14a)$$

$$\dot{m} = n\dot{m}_o \qquad (2\text{-}14b)$$

$$a = \frac{nf_o}{m} = \frac{-n\dot{m}_o c}{m} \qquad (2\text{-}15)$$

$$p = np_o = \frac{-n\dot{m}_o c^2}{2} \qquad (2\text{-}16)$$

where n is the fraction of thrusters in use and f_o, $-\dot{m}_o$ and p_o are the maximum values of thrust, propellant flow rate and exhaust power, respectively.

By eliminating the exhaust velocity, equations (2-14) through (2-16) may be manipulated into the form

$$\frac{-n\dot{m}_o}{m^2} = \frac{f_o a}{2p_o m} \qquad (2\text{-}17)$$

Integrating and expressing the result in terms of the mass ratio, one obtains

$$MR - 1 = \frac{m_o}{2p_o} \int_o^{t_f} \frac{f_o}{m} \, a \, dt \qquad (2\text{-}18)$$

Observe that the factor f_o/m is the maximum acceleration possible for a given mass. Therefore the acceleration integral for a thrust-limited vehicle is proportional to the integral over the thrusting time of

the product of the maximum instantaneous acceleration and the instantaneous acceleration. Designate this integral J^* and observe that for a vehicle of specified power and initial mass, the mass ratio is a minimum when J^* is a minimum

$$J^* = \frac{1}{2} \int_0^{t_f} a_{max}\, a\, dt \tag{2-19}$$

$$MR - 1 = \frac{\alpha}{\beta}\, J^* \tag{2-20}$$

Equations (2-19) and (2-20) correspond to equations (2-9) and (2-12) respectively, and permit a direct comparison of the performance of a hypothetical vehicle with given power and given initial mass when controlled as a VSI vehicle and then as a CSI vehicle.

The comparison is made by finding an optimal trajectory for a specified mission for each type of control and then comparing the mass ratios or the acceleration integrals for the two cases. This has been done for a simulated mission to Mars using techniques described in later chapters, and for the simple case of a transfer in field-free space.

2.6 Comparison of CSI and VSI Control

From a qualitative point of view, finding an optimal trajectory for the VSI machine consists of finding that trajectory which minimizes propellant expenditure for a given constant exhaust power but with unconstrained exhaust velocity. In the case of the CSI vehicle, the problem is that of finding the trajectory which minimizes the propellant expenditure subject to a given maximum exhaust power and a given constant exhaust velocity.

If both vehicles have the same maximum exhaust power and both control programs optimize propellant consumption then constant-specific-impulse is but a limiting case of variable-specific-impulse and cannot result in less propellant usage. Therefore the acceleration integral for a VSI transfer can be used as a reference for assessing the additional propellant cost of CSI transfers.

In order to compare the two vehicles for a transfer in the gravitational field it is necessary to use machine computation. However, they may be compared analytically in field-free space. The results are shown to provide reasonable approximations for the values obtained in the gravitational field.

Assume that in field-free space it is desired to traverse the distance L in the time T such that the rocket begins and ends at rest. This problem is solved in the literature for VSI spacecraft[34]. The solution is rederived in Appendix A and the solution for the CSI spacecraft is also derived.

The results for variable-specific-impulse thrusting are:

$$J_{min} = \frac{6L^2}{T^3} \tag{2-21}$$

$$a_o = \frac{6L}{T^2} \tag{2-22}$$

where a_o is the initial acceleration. The optimum acceleration program starts at a_o and decreases linearly with time such that $a(T) = - a_o$. (See Figure 2-c.)

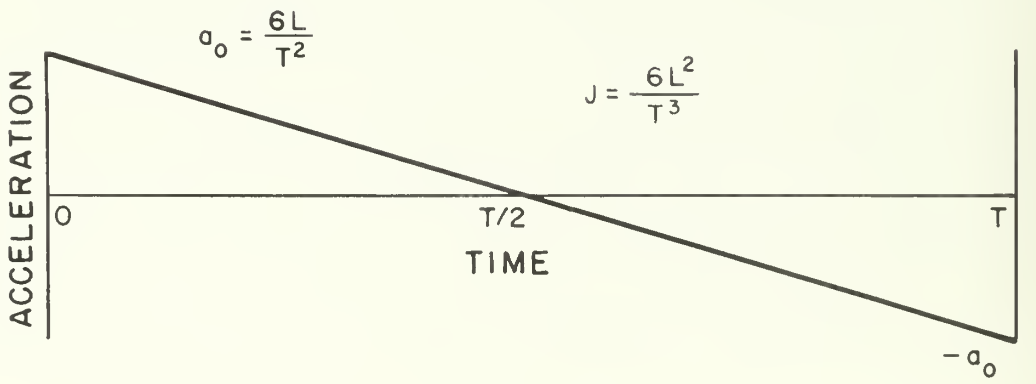

Fig. 2-c. Optimum acceleration for constant power rocket in field-free space.

For a normalized mass (i. e. m_o is unity), equation (2-12) may be written in this case as:

$$MR = \frac{6L^2}{pT^3} + 1 \tag{2-23}$$

Define the parameter R.

$$R = \frac{3J}{p} \tag{2-24}$$

$$MR = \frac{R}{3} + 1 \tag{2-25}$$

Therefore in this case

$$R = \frac{18L^2}{pT^3} \tag{2-26}$$

Equation (2-25) is plotted in Figure 2-d. R is an excellent measure of the difficulty of a mission in field-free space. For example: A large transfer distance, a short transfer time and a small energy source would produce a large value of R. Such conditions indeed represent a difficult transfer.

It should be noted that the mass ratio is linear in R. Thus VSI transfers characterized by a finite R require finite mass ratios.

For a constant-specific-impulse vehicle of identical power to perform this mission requires a specification of exhaust velocity or, more conveniently, initial acceleration. In Figure 2-d, the mass ratio is plotted as a function of R for two values of initial acceleration. The first corresponds to $a_o = \frac{6L}{T^2}$ and is asymptotic to R = 20. The second, which corresponds to an optimum value of initial acceleration for CSI machines is exponential in R.

The optimum initial acceleration for CSI vehicles is

$$a_{o(opt)_{CSI}} = \frac{2}{R} (\ln MR)\, a_{o(opt)_{VSI}} \tag{2-27}$$

For small values of R such that the mass ratio is near unity, equation (2-27) may be simplified to

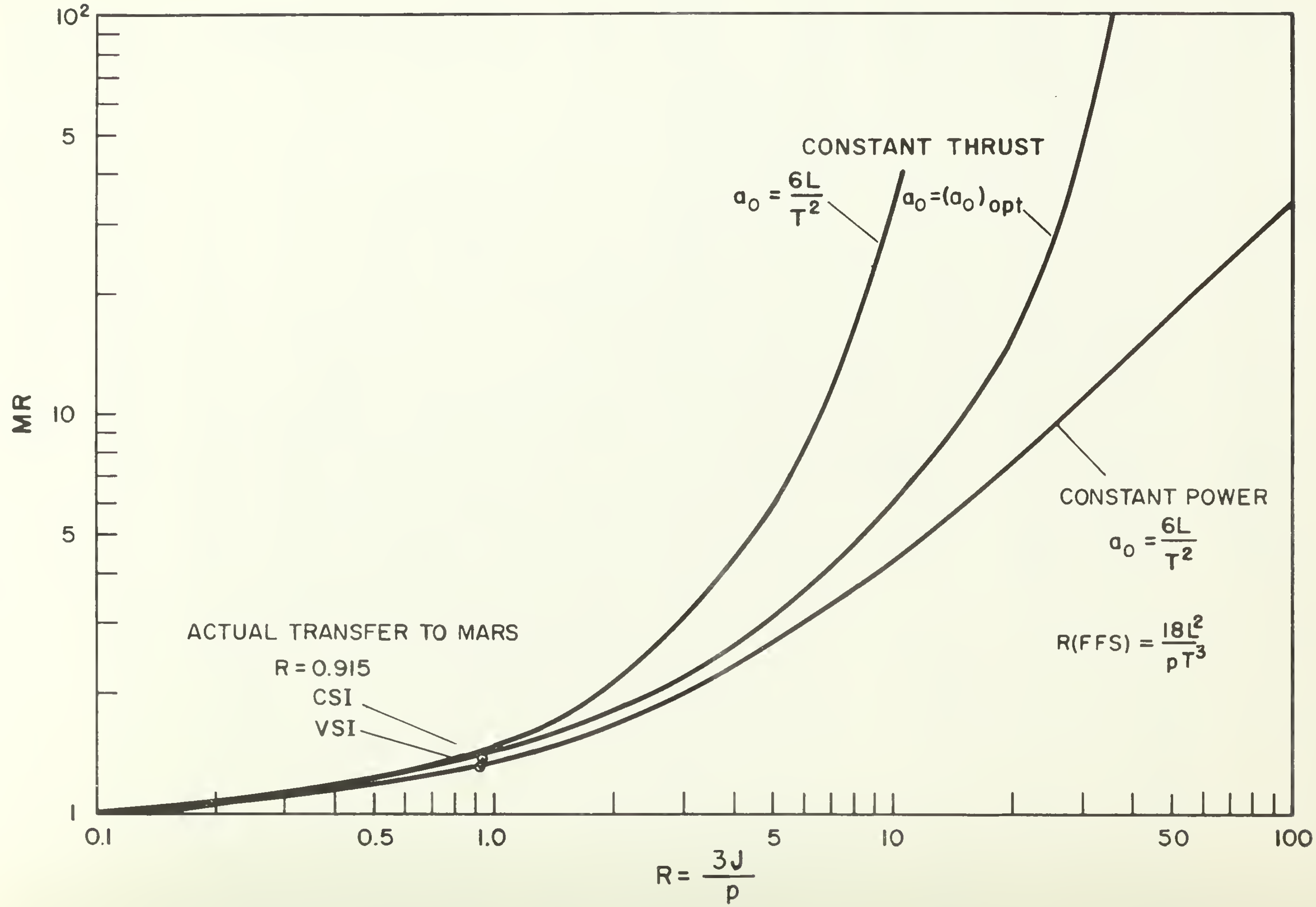

Fig. 2-d. Optimum point-to-point transfers in field-free space.

$$a_{o(opt)_{CSI}} = \frac{3}{4} a_{o(opt)_{VSI}} \tag{2-28}$$

The transcendental equations for this example are derived and solved in Appendix A.

The conclusions that may be drawn from the field-free space example are:

1) Control of thrust by varying specific impulse but with constant exhaust power is the propellant-optimal method of control.

2) For missions represented by small values of the parameter R, (less than 5) the propellant penalty for using an optimized constant specific impulse is less than 15%.

3) For large values of R the propellant cost of constant-specific-impulse is very large, even for an optimized CSI.

4) For sufficiently long transfer times the value of R may be decreased so that the propellant penalty of constant-specific-impulse is very small. (As will be subsequently shown, an optimized one way trip to Mars corresponds to an R of approximately 1).

It is not readily apparent that the results derived for field-free space are directly applicable to transfers in the gravitational fields of the solar system. However, Melbourne and Sauer[34] have computed the VSI acceleration requirements for a number of interplanetary transfers and found the values to be in surprisingly good agreement with the field-free space analysis. Therefore, it seems valid to argue on the basis of field-free space when comparing CSI and VSI systems. The numerical results in this study support this conclusion.

The parameter R, which includes the effects of both mission requirements and spacecraft power, was found to be more useful in the comparison of CSI and VSI systems than the acceleration integral alone. Some significant facts which are evident when R is used as a parameter but which are missed when the acceleration integral alone is used are:

1) increasing the spacecraft power for a particular mission reduces the propellant penalty of CSI operation,

2) the optimum initial acceleration, and thus the optimum exhaust velocity for CSI vehicles may be determined to reasonable accuracy from analysis of the VSI spacecraft using equation (2-27).

2.7 Maximizing the Payload

In section 2.6, the problem of maximizing the mass ratio is considered. In this section the energy source which maximizes the payload for a given mission and mode of operation is computed.

Both CSI and VSI vehicles are characterized by the general equation:

$$MR = \frac{\alpha}{\beta} J + 1 \tag{2-29}$$

where the term $\frac{\alpha}{\beta} J$ may be written alternatively as $\frac{m_o}{p} J$.

If the initial mass of the vehicle is considered to consist of only three parts: payload, propellant and power source, then the mass distribution may be written as

$$\frac{m_L}{m_o} = 1 - \beta - (1 - \frac{1}{MR}) \tag{2-30}$$

$$\frac{m_L}{m_o} = \frac{1}{MR} - \beta \tag{2-31}$$

where $\frac{m_L}{m_o}$ is the payload fraction.

Substituting equation (2-29) into equation (2-31), an expression for payload fraction is obtained in terms of α, β, and J.

$$\frac{m_L}{m_o} = \beta \left(\frac{1}{\alpha J + \beta} - 1 \right) \tag{2-32}$$

Differentiating equation (2-32) with respect to β and setting the derivative equal to zero results in the optimal mass distribution for either CSI or VSI space vehicles. The result is:

$$\left(\frac{m_L}{m_o}\right)_{opt} = (1 - \sqrt{\alpha J})^2 \tag{2-33}$$

$$\beta_{opt} = (\alpha J)^{1/2} - \alpha J \tag{2-34}$$

$$\left(\frac{m_p}{m_o}\right)_{opt} = (\alpha J)^{1/2} \tag{2-35}$$

where $\frac{m_p}{m_o}$ is the propellant fraction.

From the alternative forms of equation (2-29) and equation (2-34) the optimum exhaust power for a space mission is obtained.

$$\frac{\beta_{opt}}{\alpha} = \frac{P_o}{m_o} = \left(\frac{J}{\alpha}\right)^{1/2} - J \tag{2-36}$$

For the numerical work in this thesis an optimistic, but not unreasonable, value of α was selected. In all subsequent computations, an α of 10 kilograms per kilowatt is assumed. From the digital computer studies of a 150 day Earth-Mars one way transfer, a value of J was obtained for the sizing relationships. This value of J was multiplied by the factor 4 and rounded off to provide a more realistic value for a round trip to Mars. The use of an approximation is justified in that the purpose here is to provide a reasonable value of power for comparing VSI and CSI operations without placing undue emphasis on optimizing the round trip mass distribution. That is, the author is more concerned with optimal guidance of a space vehicle of given power and mode of operation than in determining whether the spacecraft is the best vehicle for the mission. Further, the preceding method of maximizing payload, although widely used in low-thrust studies, does not adequately treat a round trip mission comprised of several phases such as escape from Earth, transfer, capture at the target, a waiting period,

and similar return phases. Each of the phases may impose particular requirements which are not compatible with the other phases. For example, it may be impractical to use 100 or more days to effect a spiral escape even though the optimized solution requires it. Consequently the derivation in this section must be considered as only an approximate method of maximizing payload for realistic round trip mission planning.

The value of J selected was 10^{-6}. The units are (astronomical units)^2 per (day)^3. Inserting these values of a and J (in compatable units) into the equation (2-36), a value for p_o/m_o was computed for use in the numerical work.

$$\frac{p_o}{m_o} = 0.0242 \text{ kw/kg} = 0.6988 \times 10^{-6} (\text{A. u.})^2/(\text{day})^3 \qquad (2\text{-}37)$$

Using the value obtained in equation (2-37), values of MR were extracted from computer results and compared with the field-free space values for both CSI and VSI engine control. The results of this comparison are plotted on the curves of Figure 2-d. Agreement is good for the cases tested and warrants further investigation for different values of the parameter R.

Values of initial acceleration and specific impulse which are characteristic of the values obtained in the numerical studies are

$$a_o/g_o = 1.2 \times 10^{-4} \qquad (2\text{-}38)$$

$$I_{sp} = 4000 \text{ seconds} \qquad (2\text{-}39)$$

CHAPTER III

LINEAR GUIDANCE

3.1 Summary of Chapter III

The guidance problem for continuous-thrust vehicles is formulated in this chapter using perturbation techniques and the theory of optimum control. A reference trajectory which satisfies mission requirements is assumed to exist and its associated control program is known. The consequences of allowing the reference trajectory to be an optimal trajectory are examined. Linearized equations of state are solved together with their adjoint set to produce a state transition equation applicable for small perturbations around the reference trajectory. The state transition equation is then shown to be suitable as a guidance equation. Solutions for the guidance equation are derived for fixed-time-of-arrival (FTA) guidance. Variable-time-of-arrival (VTA) guidance is solved for certain restricted cases.

3.2 General Remarks

In terms of the discussion in Chapter II, the ensuing derivation of linear guidance may be characterized as a method of minimizing the acceleration integral, J, between the present position and the target position. For this purpose it is convenient to abandon temporarily the engineering aspects of low-thrust transfers and to consider the problem in terms of the calculus of variations and the theory of optimum control.

The guidance schemes suggested for interplanetary vehicles following ballistic trajectories usually require that midcourse corrections

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state transition equation: An equation which expresses the state of the vehicle at any given time in terms of 1) the state at any other time and 2) the control existing between the two times.

guidance equation: An equation, possessing a solution which satisfies the mission requirements for the spacecraft. The solution of a guidance equation is often termed the "control law" or "control program" for the physical guidance system.

minimize, subject to one or more constraints, the position and velocity variations at the destination point.

The characteristics of a continuous-thrust vehicle permit a guidance scheme which will null the position and velocity variations at the destination, provided guidance is begun sufficiently early in the transfer. For all practical vehicles, if guidance is not initiated early in the transfer the low-thrust vehicle will arrive at some point, beyond which insufficient power is available to complete the mission. The vehicle is then said to be "in extremis". This boundary may be visualized as a conical surface surrounding the reference trajectory with its apex at the target point. The "distance" in phase space from the trajectory to the surface depends upon the excess power allowed for guidance. The problem of determining the optimum power and propellant allowance for guidance is not studied in this report; however, the techniques of this chapter are readily applicable to such investigations and are recommended as a tool for future work. In this chapter it is assumed that the reserve power is adequate for satisfying the solution to the guidance equation.

3.3 Formulation of the Problem

The basis for the subsequent analysis is that a vehicle is in transit between two planets. The control program in use corresponds to some known reference trajectory. Measurement of vehicle state indicates that the vehicle is off the reference trajectory by some small amount.

The first problem of interest asks the question: "What is the effect of the state variation at the time of the measurement on the vehicle state at any future time?" The next obvious question is: "What is the new control program which will cause the vehicle to satisfy mission requirements?" The third is: "If there is more than one control program available, what criterion should be used for selecting one of them?"

The third question may be disposed of immediately. The thesis is concerned with propellant-optimal guidance. Therefore the choice between controls is on the basis of minimum propellant consumption. From Chapter II, a criterion which satisfies this objective is the

acceleration integral There may be other criteria which are equally as good, however J was chosen because it is easy to manipulate and is easily interpreted in the physical problem.

The second question requires that the mission be defined. For this purpose it is sufficient to require that the final position and velocity of the vehicle have certain well defined values.

Depending upon the particular mission, the above guidance objectives may be satisfied at the preplanned arrival time (FTA guidance) or at some time different from the planned time but consistent with the mission (VTA guidance). The FTA problem is readily solved by the linear methods of this chapter. The VTA propellant-optimal problem is considerably more difficult. A method of obtaining restricted solutions by linear methods is presented and discussed

3.4 Selection of State Variables

The differential equations of motion for celestial bodies admit six constants of integration. The selection of the six quantities to represent this motion is to some extent arbitrary. To be consistent with the definition of mission requirements in this investigation, and because they are convenient, the three components of position and the three components of velocity are chosen. The convenience is due to their facility for visualization and their relative facility for physical measurement.

Since the conservation of propellant is important, an additional differential equation describing the mass change due to propellant flow must be included for certain cases. Thus seven independent, but coupled, variables are sufficient to describe the state of the vehicle at any time. The state will be subsequently written as a vector which has components of position, velocity and mass.

$$\underline{s} = \begin{Bmatrix} \underline{r} \\ \underline{v} \\ m \end{Bmatrix} \tag{3-1}$$

The second order differential equations of motion will be rewritten as first order equations to conform to the above definition of state.

For this thesis, only central force fields are considered. It is subsequently shown that extension to more general gravitational fields does not invalidate the theory but does introduce computational complexities which serve only to obscure the physical concept if introduced prematurely.

With this introduction the differential equations of state are now presented. A nonrotating frame with its origin at the central body is assumed.

$$\dot{\underline{s}} = \underline{g}\,(\underline{s},\ \underline{a}) \tag{3-2}$$

where

$$\underline{g} = \begin{Bmatrix} \underline{v} \\ -\dfrac{\mu}{r^3}\,\underline{r} + \underline{a} \\ g_m\,(m, a) \end{Bmatrix} \tag{3-3}$$

and where $\underline{a}$ is the thrust acceleration, subsequently called the control vector, μ is the gravitation constant for the central body, and $g_m(m,a)$ is the mass rate equation for the particular vehicle under consideration. From Chapter II

$$g_m(m,a) = -\frac{a^2 m^2}{2p} \quad \text{(VSI)} \tag{3-4}$$

$$g_m(m,a) = -\frac{am}{c} \quad \text{(CSI)} \tag{3-5}$$

Throughout the report the variable m is a normalized mass, that is $m_o = 1$. This obviates the necessity of selecting a value for total vehicle mass. The power and the propellant flow are likewise normalized variables. That is

$$p = \frac{\text{Exhaust power}}{\text{initial mass}} \tag{3-6}$$

$$\dot{m} = \frac{\text{propellant flow rate}}{\text{initial mass}} \tag{3-7}$$

3.5 Linearization of the Equation of State

The assumptions of a known reference trajectory, known control program and of small deviations from the reference trajectory permit the application of linear perturbation theory. Thus

$$\delta \dot{\underline{s}} = \frac{\partial \underline{g}}{\partial \underline{s}} \; \delta \underline{s} + \frac{\partial \underline{g}}{\partial \underline{a}} \; \delta \underline{a} \tag{3-8}$$

From equation (3-8) the matrices A and B are defined.

$$A = \frac{\partial \underline{g}}{\partial \underline{s}} = \begin{bmatrix} O_3 & I_3 & \underline{O} \\ -G & O_3 & \underline{O} \\ \underline{O}^T & \underline{O}^T & \dfrac{\partial g_m}{\partial m} \end{bmatrix} \tag{3-9}$$

$$B = \frac{\partial \underline{g}}{\partial \underline{a}} = \begin{bmatrix} O_3 \\ I_3 \\ \dfrac{\partial g_m}{\partial \underline{a}} \end{bmatrix} \tag{3-10}$$

where

$$G = \frac{\partial}{\partial \underline{r}} \left(\frac{\mu}{r^3} \, \underline{r} \right) \tag{3-11}$$

$$G = \frac{\mu}{r^5} \left(r^2 I - 3 \, \underline{r} \, \underline{r}^T \right) \tag{3-12}$$

$$\frac{\partial g_m}{\partial m} = - \frac{a^2 m}{p} \quad \text{(VSI)} \tag{3-13}$$

$$\frac{\partial g_m}{\partial m} = - \frac{a}{c} \quad \text{(CSI)} \tag{3-14}$$

$$\frac{\partial g_m}{\partial \underline{a}} = - \frac{m^2 \, \underline{a}^T}{p} \quad \text{(VSI)} \tag{3-15}$$

$$\frac{\partial g_m}{\partial \underline{a}} = - \frac{m}{c} \; \frac{\underline{a}^T}{a} \quad \text{(CSI)} \tag{3-15a}$$

The matrix G is a symmetric matrix of gravitational gradients. The extension to a general gravitational field does not change this property[31]. However, numerical integration in such a field requires an ephemeris for the disturbing bodies. This is a computational problem and does not affect the theory.

Equations (3-8) through (3-15a)treat the variations in state velocity as functions of the variations in state and variations in the control vector, $\underline{a}$. It is often desirable to study the effects of variations in performance parameters such as power, exhaust velocity, propellant flow, etc. For this purpose one may rewrite the vector $\underline{a}$ and the propellant flow equation in terms of the appropriate parameters. New matrices A and B will be formed for this purpose. These cases are derived in Appendix C since they do not contribute to the discussion of guidance. However, occasionally in the guidance problem it is desirable to work with the thrust, $\underline{f}$, as the control instead of $\underline{a}$. The theory subsequently presented is applicable to this formulation also. For the present, however, the form

$$\delta\dot{\underline{s}} = A\delta\underline{s} + B\delta\underline{a} \tag{3-16}$$

is considered to be the fundamental formulation for the guidance problem.

With the perturbed equations of state, as defined by (3-16) , and using the adjoint method, it is possible to derive a state transition equation.

3.6 Method of Adjoints

One description for the method of adjoints is obtained by considering a set of equations related to (3-16) by the matrix equation

$$\dot{\Lambda} = - \Lambda A \tag{3-17}$$

Equation (3-17) , with arbitrary boundary conditions, is said to be adjoint to (3-16) and the elements of Λ are the adjoint variables. If equation (3-16) is premultiplied by Λ , (3-17) post-multiplied by $\delta\underline{s}$ and the resulting equations are added, then one obtains

$$\frac{d}{dt} (\Lambda \delta\underline{s}) = \Lambda B\delta\underline{a} \tag{3-18}$$

Along a given reference trajectory, i. e $\delta \underline{a} = \underline{O}$, equation (3-18) has the solution

$$\Lambda \delta \underline{s} = \text{constant vector} \tag{3-19}$$

If (3-17) is integrated along the trajectory, subject to suitable boundary conditions, then Λ is a known function of time and the state variation at any time may be determined from the state variation at any other time, provided the reference control program is used.

The adjoint method may be described as the process of introducing a set of known auxiliary variables, Λ, which satisfy (3-17) and which transform the state variation at a given time into the state variation at any other time, provided a forcing function does not exist.

A scalar approach to adjoint equations and other examples of their application are presented in Appendix B.

A useful system of equations closely related to the adjoint set is a set often called the fundamental set or fundamental solution. It satisfies the relation

$$\dot{\Phi} = A \Phi \tag{3-20}$$

Combining (3-20) and (3-17) by the appropriate pre-and post-multiplication, adding and integrating, yields the first integral

$$\Lambda \Phi = \text{constant matrix} \tag{3-21}$$

If the boundary conditions are chosen such that

$$\Lambda(t_f) = I \tag{3-22}$$

$$\Phi(0) = I \tag{3-23}$$

then

$$\Lambda \Phi = \Lambda(0) = \Phi(t_f) \tag{3-24}$$

This relationship is useful in later discussions.

3.7 The State Transition Equation

If equation (3-18) is integrated between the times t_1 and t_2, the result is

$$\Lambda_2 \delta \underline{s}_2 = \Lambda_1 \delta \underline{s}_1 + \int_{t_1}^{t_2} \Lambda B \delta \underline{a} \, dt \tag{3-25}$$

Then, provided Λ_2 is nonsingular

$$\delta \underline{s}_2 = \Lambda_2^{-1} \Lambda_1 \delta \underline{s}_1 + \Lambda_2^{-1} \int_{t_1}^{t_2} \Lambda B \, \delta \underline{a} \, dt \qquad (3\text{-}26)$$

A proof is presented in Appendix C that Λ^{-1} is not singular.

The product $\Lambda_i^{-1} \Lambda_j$ is called the state transition matrix and is denoted by T_{ij}, where i and j represent any two times. Equation (3-26) is called the state transition equation. It transforms the state variation at t_1 and the control perturbation between t_1 and t_2 into the state variation at t_2.

There are three useful applications of the state transition equation:

1) It may be used as a tool for studying the sensitivity of the trajectory to perturbations in launch conditions and to anomalies in engine performance. Friedlander[10] has performed some investigations in this area. This application is not pursued further in this report except for derivation of applicable formulations in Appendix C.

2) If engine performance is measured by an accelerometer or other device which can be related to the state equations, then the state transition equation may be used in the navigation of a spacecraft by relating the measured engine performance to the state of the vehicle. This application is discussed in Chapter IV.

3) The most important application of equation (3-26) is its use in guidance. If, in (3-26) , t_2 is considered as the final time, then the state and control errors occuring along the trajectory, may be related to the final state variation. Since the mission objective may be defined by the final position and velocity vectors, (3-26) meets the requirements for a guidance equation provided that, for any state variation $\delta \underline{s}_t$ it is possible to find a solution $\underline{a} + \delta \underline{a}$ which will cause the vehicle to match the physicalboundary conditions at the terminal point. For this application the terminology "guidance equation" is preferable to "state transition equation" and will be used to define the

equation when t_2 is the final time and the boundary conditions on the adjoint variables are chosen as $\Lambda(t_f) = I$. Thus the guidance equation is

$$\delta \underline{s}_f = \Lambda_t \, \delta \underline{s}_t + \int_t^{t_f} \Lambda B \, \delta \underline{a} \, dt \tag{3-27}$$

where the initial time t is any time between the launch time and final time, at which a new control program is desired.

The integrand in (3-26) and (3-27) is observed to be quite simple when the matrices are partitioned and expanded. For the control vector $\underline{a}$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{O} \\ \Lambda_{21} & \Lambda_{22} & \underline{O} \\ \underline{O}^T & \underline{O}^T & \Lambda_{33} \end{bmatrix} \tag{3-27a}$$

Using (3-10) and expanding

$$\Lambda B = \begin{bmatrix} \Lambda_{12} & \\ \Lambda_{22} & \\ \Lambda_{33} & \dfrac{\partial g_m}{\partial \underline{a}} \end{bmatrix} \tag{3-27b}$$

where Λ_{33} is a scalar and the remaining Λ's are three by three matrices. The vectors are three component null vectors.

3.8 Solution of the FTA Guidance Problem

The arguments presented in this section are equally valid whether the desired solution is the acceleration program $\underline{a} + \delta\underline{a}$ or a thrust program $\underline{f} + \delta\underline{f}$. A discussion of the requirements for and the consequences of interchanging control vectors is presented in Appendix C. Since the propellant expenditure is so easily written in terms of the acceleration integral, especially for VSI vehicles, it is computationally convenient to work with $\underline{a}$ as the control. If $\underline{a}$ is used in solving the guidance problem, the set of adjoint functions, Λ, may be reduced immediately from a seven by seven matrix to a six by six matrix since the equations of motion, (3-3), do not contain mass explicitly and the "mission" is to

satisfy terminal conditions on the equations of motion. However, when working with the CSI vehicle it is usually more convenient to use the thrust, $\underline{f}$. Acceleration is used for both vehicles in this report in order to be consistent. Since thrust and acceleration are related through mass, the equations of motion using thrust as a control must contain the state variable, m. For this reason the adjoint set cannot be reduced a priori to a six by six matrix. In recognition of this fact and since both control vectors are useful, the Λ matrix is always considered to be a seven by seven matrix; with the understanding that if $\underline{a}$ is the control, certain elements are identically zero. This does not present any problem in manipulation and need not be considered further except when Λ and B are to be evaluated numerically. In the subsequent discussion, however, a reduced adjoint set is introduced and is denoted by Λ^*. This reduction is not associated with the difference between acceleration and thrust formulations. Both a Λ matrix and a Λ^* matrix exist for each control vector.

From equation (3-27) it is clear that if a state variation exists at time t, then in the absence of control changes, a variation in final state will occur which is given by

$$\delta \underline{s}_f = \Lambda_t \, \delta \underline{s}_t \tag{3-28}$$

The guidance criterion is that the final position and velocity must satisfy mission requirements. Presumably the reference trajectory satisfies these requirements. Thus the position and velocity variations from the reference must be zero at the final time if FTA guidance is used. Stated mathematically, it is required that

$$\delta \underline{s}_f = \begin{Bmatrix} \underline{O} \\ \underline{O} \\ \delta m_f \end{Bmatrix} \tag{3-29}$$

where δm_f is a small but unknown variation in final mass. Whatever the value of δm_f, it is of no immediate concern. To reflect this it is convenient to drop the δm_f and define the "miss" at the target which arises from the state variation at time t as

$$\underline{\xi}_t = \Lambda_t^* \, \delta \underline{s}_t \tag{3-30}$$

where $\underline{\xi}$ is a six component vector of position and velocity variations and Λ^* is a six by seven matrix obtained by deleting the seventh row of Λ. Clearly δm_f can be obtained, once the corrected control is known, from

$$\delta m_f = \underline{\Lambda}_{7t}^T \, \delta \underline{s}_t + \int_t^{t_f} \underline{\Lambda}_7^T \, B \, \delta \underline{a} \, dt \tag{3-31}$$

where $\underline{\Lambda}_7^T$ denotes the deleted seventh row of the adjoint set.

Since the state error $\delta \underline{s}_t$ is assumed to be small, so that linear theory is valid, it follows that a small correction to the reference control program will be sufficient to null the miss vector in the remaining flight time. The foregoing is true provided the vehicle has sufficient thrust. That is, it is not "in extremis". Assume it is not. Then the miss may be reduced at a rate such that

$$\dot{\underline{\xi}} = \Lambda^* B \, \delta \underline{a} \tag{3-32}$$

or by integrating, such that

$$\underline{O} - \underline{\xi}_t = \int_t^{t_f} \Lambda^* B \, \delta \underline{a} \, dt \tag{3-33}$$

That the control $\delta \underline{a}$ is not unique, except for a $\underline{\xi}_t$ which requires maximum thrust continuously, is evidenced by the fact that for ballistic guidance, ideally, only two corrections are needed to null position and velocity error; a midcourse correction to correct position and a terminal correction for velocity. Thus for continuous thrusting vehicles an infinity of solutions exists. The criterion for selecting a unique control has already been given; the acceleration integral J or J^* must be minimum. Mathematically, the problem for VSI vehicles may be stated: It is required that

$$\left\{ \begin{matrix} \underline{O} \\ \underline{O} \end{matrix} \right\} = \underline{\xi}_t + \int_t^{t_f} \Lambda^* B \, \delta \underline{a} \, dt \tag{3-34}$$

and that J is a minimum, where

$$J = \int_t^{t_f} \frac{(\underline{a} + \delta \underline{a})^T (\underline{a} + \delta \underline{a})}{2} \, dt \tag{3-35}$$

For convenience define the optimal control, $\underline{a}^o$

$$\underline{a}^o = \underline{a} + \delta\underline{a} \tag{3-36}$$

For a VSI vehicle with no constraints on $\underline{a}^o$ the set (3-34) and (3-35) can be solved by a direct application of the calculus of variations. The vector space around the reference trajectory is explicitly assumed to be flat to first order. Consequently Λ and B are invariant between neighboring trajectories. Thus

$$\delta\left(\int_t^{t_f} \frac{\underline{a}^{oT}\,\underline{a}^o}{2}\,dt + \underline{\pi}^T\left\{\underline{\xi} + \int_t^{t_f} \Lambda^* B(\underline{a}^o - \underline{a})\,dt\right\}\right) = 0 \tag{3-37}$$

where $\underline{\pi}$ is a vector of constant Lagrange multipliers and the subscript on $\underline{\xi}_t$ is dropped. Expanding (3-37) one obtains

$$\underline{a}^{o^T} = -\underline{\pi}^T \Lambda^* B \tag{3-38}$$

The vector $\underline{\pi}$ is eliminated with help of equation (3-33).

$$\left\{\begin{matrix}\underline{O}\\ \underline{O}\end{matrix}\right\} = \underline{\xi} - \int_t^{t_f} \Lambda^* B\left\{B^T \Lambda^{*T}\underline{\pi} + \underline{a}\right\} dt \tag{3-39}$$

$$-\underline{\pi} = M^{-1}(\underline{\eta} - \underline{\xi}) \tag{3-40}$$

where

$$M = \int_t^{t_f} \Lambda^* B\, B^T \Lambda^{*T}\, dt \tag{3-41}$$

$$\underline{\eta} = \int_t^{t_f} \Lambda^* B\, \underline{a}\, dt \tag{3-42}$$

Thus

$$\underline{a}^o = B^T \Lambda^{*T} M^{-1}\left(\underline{\eta} - \underline{\xi}\right) \tag{3-43}$$

The solution (3-43) is valid provided the reference trajectory is sufficiently close to an optimal that $\delta\underline{a}$ is small. It is unique provided M is not singular. A proof is presented in Appendix C for the existence of M^{-1}.

A physical interpretation of the quantities M and $\underline{\eta}$, defined by (3-41) and (3-42), is valuable in understanding the solution. First observe that $\underline{\xi}$ has two interpretations. From (3-30), $\underline{\xi}$ is the miss

arising from a state variation, $\delta \underline{s}_t$, when the reference control is used. From (3-33), $-\underline{\xi}$ may be interpreted as the miss that results from a control variation, $\delta \underline{a}$, occuring after time t but with $\delta \underline{s}_t = \underline{0}$. Similarly from (3-42), $-\underline{\eta}$ is interpreted as the particular miss that arises if $\delta \underline{a} = -\underline{a}$; that is, coast is initiated. If the miss, $\underline{\xi}$, due to a state variation and the miss, $-\underline{\eta}$, due to initiating coast are such that $(\underline{\xi} - \underline{\eta}) = \underline{0}$, the optimal control is the null vector. Although such a solution will generally violate linearity assumptions, it will indeed result in a minimum J, namely zero.

If the product $\Lambda^* B$ is interpreted as the sensitivity of the final miss to a unit impulse in each component of the control at time t, then M may be considered as a weighted total sensitivity of the miss to a unit control applied continuously between t and t_f. In a physical sense the procedure computes the vector sum of the miss using the reference control vector and the miss using a null control vector; then selects the control at each point according to the sensitivity of the miss at that point.

It is interesting to note the solution which results if, instead of J, the minimization criterion is

$$S = \int_t^{t_f} \frac{\delta \underline{a}^T \, \delta \underline{a}}{2} \, dt \tag{3-44}$$

Proceeding as in equations (3-37) through (3-41) but solving for $\delta \underline{a}$, the result is

$$\underline{a}^o = \underline{a} + \delta \underline{a} = \underline{a} - B^T \Lambda^{*T} M^{-1} \underline{\xi} \tag{3-45}$$

Comparing (3-45) with (3-43) it is observed that for $\underline{a} = B^T \Lambda^{*T} M^{-1} \underline{\eta}$ the results are the same. The preceding expression for $\underline{a}$ satisfies (3-42) therefore for small variations around the reference trajectory the two methods result in the same control. This is only true for VSI trajectories, however.

Before proceeding to consider CSI solutions it is desirable to examine the consequences of the reference trajectory being optimal for the initial launch point. If the reference is initially optimal but a

state variation occurs such that a miss, $\underline{\xi}$, will result from use of the reference control program, the reference control ceases to be an optimum because the boundary conditions are not satisfied. Also the particular formulation used for the problem does not result in a singular M matrix as is often the case. Therefore, in this study, the only consequence of using an optimal reference is the assurance that $\delta\underline{a}$ will not violate linearity assumptions.

A new approach is now presented which can be used for both VSI and CSI control. To illustrate this method the VSI problem is solved again but with a constraint.

Assume that thrust is not to exceed an amount f_o. Therefore thrust acceleration cannot exceed f_o/mm_o. To be consistent with the normalized mass variable, m, used in this report denote f_o/m_o as the initial acceleration limit a_o. The constraint may now be written as

$$a \leq \frac{a_o}{m} \tag{3-46}$$

Following Kalman[16] and using Pontryagin's principle (Appendix D), form the scalar Hamiltonian, H

$$H = \frac{\underline{a}^{o^T} \underline{a}^o}{2} + \underline{\nu}^T \dot{\underline{\xi}} \tag{3-47}$$

The theory states that if the cost, $\int ((a^o)^2/2) dt$, is to be a minimum, then for each point along the trajectory H must be a minimum, where $\underline{\nu}$ is an unknown vector, often called the costate, which satisfies the adjoint relationship, and $\dot{\underline{\xi}}$ is the state velocity. The linearization in this chapter permits considerable simplification. Using (3-32), $\underline{\xi}$ is interpreted as a variable representing the velocity of the final state. The variable $\underline{\xi}_t$ is the initial condition for $\dot{\underline{\xi}}$ and is a function of the lower limit of integration, t. $\underline{\xi}_t$ is evaluated from the vehicle state, $\delta\underline{s}_t$, using (3-30). Because $\underline{\xi}$ is a variable in state space at the final point, $\underline{\nu}$ is a constant vector. In particular it is the final value of the general time varying costate vector.

Therefore, using (3-36)

$$\dot{\underline{\xi}} = \Lambda^* B (\underline{a}^o - \underline{a}) \tag{3-48}$$

Further

$$H = \frac{\underline{a}^{o^T} \underline{a}^o}{2} + \underline{\nu}^T \Lambda^* B (\underline{a}^o - \underline{a}) \qquad (3\text{-}49)$$

$$H = -\underline{\nu}^T \Lambda^* B \underline{a} + \frac{\underline{a}^{o^T} \underline{a}^o}{2} + \underline{\nu}^T \Lambda^* B \underline{a}^o \qquad (3\text{-}50)$$

The first term in (3-50) is a constant independent of $\underline{a}^o$, thus H is a minimum when the sum of the last two terms, H_1, is a minimum.

The control $\underline{a}^o$ is to be determined such that H_1 is a minimum.

$$\text{minimize } H_1 = \frac{\underline{a}^{o^T} \underline{a}^o}{2} + \underline{\nu}^T \Lambda^* B \underline{a}^o \qquad (3\text{-}51)$$

$$\text{subject to} \qquad a \leq \frac{a_o}{m} \qquad (3\text{-}46)$$

$$\text{now let} \qquad \underline{\nu}^T \Lambda^* B = \underline{q}^T \qquad (3\text{-}52)$$

Since $\underline{q}$ and $\underline{a}^o$ are both three component vectors, one may be obtained from the other by a scalar multiplication and a rotation. Thus

$$\underline{a}^o = \gamma C \underline{q} \qquad (3\text{-}53)$$

where γ is a positive scalar and C a coordinate rotation. Inserting (3-53) into (3-51)

$$H_1 = \frac{\gamma^2 q^2}{2} + \gamma \underline{q}^T C \underline{q} \qquad (3\text{-}54)$$

For any value of γ, H_1 is minimum if $\underline{q}^T C \underline{q}$ has maximum magnitude and is negative. But

$$|\underline{q}^T C \underline{q}| \leq |\underline{q}| \, |C \underline{q}| \qquad (3\text{-}55)$$

where equality holds for C = I. It is apparent that H_1 is a minimum only if C = - I and $(\gamma^2/2 - \gamma)$ is a minimum.

$$H_1 = \left(\frac{\gamma^2}{2} - \gamma\right) q^2 \qquad (3\text{-}56)$$

Thus when constraint (3-46) is applied, H_1 is a minimum if and only if

$$\underline{a}^o = -\frac{a_o}{m}\frac{\underline{q}}{q} \qquad q > \frac{a_o}{m} \tag{3-57}$$

$$\underline{a}^o = -\underline{q} \qquad q \leq \frac{a_o}{m} \tag{3-58}$$

$$\gamma = \min\left(1; \frac{a_o}{mq}\right) \tag{3-59}$$

or

$$\underline{a}^o = -\gamma B^T \Lambda *^T \underline{\nu} \tag{3-60}$$

Comparing this solution with the variational solution, equation (3-38), one observes that except for the constraint γ the solutions are identical and $\underline{\nu} = \underline{\pi}$. γ may be interpreted as a switching function which carries the thrust restriction (3-46).

Having gained confidence in the Hamiltonian, now apply the method to the CSI transfer for which the classical approach is degenerate. The appropriate cost is

$$J* = \int_{t}^{t_f} \frac{a_o\, a^o}{2m}\, dt. \tag{2-19}$$

Minimize

$$H = \frac{a_o\, a^o}{2m} + \underline{\nu}^T \Lambda * B\,(\underline{a}^o - \underline{a}) \tag{3-61}$$

$$\text{subject to} \quad a \leq \frac{a_o}{m} \tag{3-46}$$

Again $\underline{\nu}^T \Lambda * B\,\underline{a}$ is a constant, therefore

$$\text{minimize } H_1 = \frac{a_o}{2m}\, a^o + \underline{q}^T \underline{a}^o \tag{3-62}$$

Where $\underline{q}$ is defined as before.

The previous argument holds, to yield

$$\underline{a}^o = -\gamma\, \underline{q} \tag{3-63}$$

and

$$H_1 = \gamma\left(\frac{a_o}{2m} - q\right) q \tag{3-64}$$

In this case H_1 is a minimum if and only if

$$\underline{a}^o = -\gamma B^T \Lambda^{*T} \underline{\nu} \tag{3-65}$$

where

$$\left\{ \begin{array}{ll} \gamma = 0 & q < \dfrac{a_o}{2m} \\ \gamma = \dfrac{a_o}{mq} & q \geq \dfrac{a_o}{2m} \end{array} \right\} \tag{3-66}$$

For the case $q = a_o/2m$, γ is actually indeterminate. However this is not an important consideration for this thesis since equality holds only for infinitesimal time periods.

Equations (3-65) and (3-66) yield the "bang-bang" solution characteristic of optimal trajectories for which the cost is a linear function of the control. Again, the solution is strictly valid only for perturbations around a reference trajectory. In order to complete the CSI solution it is necessary to evaluate the constant vector $\underline{\nu}$. The evaluation is more difficult in the CSI case than for VSI because the control program is discontinuous. $\underline{\nu}$ is evaluated by first applying the boundary conditions (3-34) and using (3-36).

$$\underline{O} = \underline{\xi} + \int_t^{t_f} \Lambda^* B (\underline{a}^o - \underline{a})\, dt \tag{3-67}$$

Assume that $\underline{a}^o$ and $\underline{a}$ differ only because their respective values of γ and $\underline{\nu}$ are different. The assumption merely implies that $\Lambda^* B$ is the same for neighboring trajectories. With this assumption rewrite (3-65) as

$$\underline{a}^o = -\gamma^o B^T \Lambda^{*T} \underline{\nu}^o \tag{3-68}$$

for the corrected control program, and let

$$\underline{a} = -\gamma B^T \Lambda^{*T} \underline{\nu} \tag{3-69}$$

represent the reference control program. Then (3-67) becomes

$$\underline{O} = \underline{\xi} - \int_t^{t_f} \Lambda^* B B^T \Lambda^{*T} (\gamma^o \underline{\nu}^o - \gamma \underline{\nu})\, dt \tag{3-70}$$

which may be rewritten as

$$\underline{\xi} = \int_t^{t_f} \Lambda^* \, B \, B^T \, \Lambda^{*T} \, \Delta(\gamma \, \underline{\nu}) \, dt \tag{3-71}$$

The increment $\Delta \, (\gamma \, \underline{\nu})$ represents the difference in the control program between the reference trajectory and the corrected trajectory.

Since γ is a function of $\underline{\nu}$, a solution may be obtained by formally differentiating and solving (3-72) for $\Delta \, \underline{\nu}$.

$$\underline{\xi} = \left[\frac{\partial}{\partial \underline{\nu}} \int_t^{t_f} \Lambda^* \, B \, B^T \, \Lambda^{*T} \gamma \, \underline{\nu} \, dt \right] \Delta \, \underline{\nu} \tag{3-72}$$

The function γ is discontinuous at switch points, which occur at times t_k in the interval t_f - t. Consequently the integral (3-72) must be separated into regions of coasting and thrusting and Leibniz's rule used for the differentiation. A term of the form

$\pm \, \Lambda^* \, B \, B^T \, \Lambda^{*T} \gamma \, \underline{\nu} \Big|_{t_k} \dfrac{\partial t_k}{\partial \underline{\nu}} \, \Delta \, \underline{\nu}$ results for each switch point, where plus is for switch off and minus is for switch on.

The term $\partial \gamma / \partial \, \underline{\nu}$ is evaluated in the continuous regions from the definitions (3-52) and (3-66). The term $\partial t_k / \partial \, \underline{\nu}$ is evaluated by considering

$$q = q \, (t, \underline{\nu}) \tag{3-73}$$

From (3-66)

$$q \, (t_k, \underline{\nu}) = \frac{a_o}{2m} \tag{3-74}$$

Differentiating (3-74) with respect to $\underline{\nu}$, one obtains

$$\frac{\partial q}{\partial t} \frac{\partial t_k}{\partial \underline{\nu}} + \frac{\partial q}{\partial \underline{\nu}} = 0 \tag{3-75}$$

$$\frac{\partial t_k}{\partial \underline{\nu}} = - \frac{\dfrac{\partial q}{\partial \underline{\nu}}}{\dfrac{\partial q}{\partial t}} \tag{3-76}$$

The derivatives of γ and q are evaluated in Chapter VI when it becomes necessary to specify the explicit form for computation.

Solving for $\Delta \underline{\nu}$, one obtains

$$\Delta \underline{\nu} = M^{*-1} \underline{\xi} \tag{3-77}$$

where

$$M^* = \int_t^{t_f} \Lambda^* B B^T \Lambda^{*T} \left[\gamma I + \underline{\nu} \frac{\partial \gamma}{\partial \underline{\nu}} \right] dt \pm \Lambda^* B B^T \Lambda^{*T} \gamma \underline{\nu} \Big|_{t_k} \frac{\partial t_k}{\partial \underline{\nu}} \tag{3-78}$$

The constant vector $\underline{\nu}^o$ is obtained from

$$\underline{\nu}^o = \underline{\nu} + \Delta \underline{\nu} \tag{3-79}$$

where

$$\underline{\nu} = - M^{-1} \underline{\eta} \tag{3-80}$$

$$M = \int_t^{t_f} \gamma \Lambda^* B B^T \Lambda^{*T} dt \tag{3-81}$$

and

$$\underline{\eta} = \int_t^{t_f} \Lambda^* B \underline{a} \, dt \tag{3-82}$$

The values for γ^o are obtained from equation (3-66) by using the computed $\underline{\nu}^o$ in q.

With all quantities defined in the preceding equations, the corrected control program is given by

$$\underline{a}^o = \gamma^o B^T \Lambda^{*T} \left(M^{-1} \underline{\eta} - M^{*-1} \underline{\xi} \right) \tag{3-83}$$

3.9 Application of the Guidance Theory

In order to use the theory of section 3.8 for the guidance of space vehicles it is necessary to compute the quantities in equation (3-43) for VSI guidance or in equation (3-83) for CSI guidance. Since (3-43) may be regarded as a special case of (3-83), only the latter is discussed.

If the reference trajectory is known, then the elements of the matrices B, Λ, M, and M* and the components of the vector $\underline{\eta}$ can be

computed for any point along the trajectory Whether these quantities are precomputed and stored prior to launch or are computed on board as needed will be determined by the state of computer technology when the vehicle is designed. This problem is not relevant to the present discussion. In either case, the first step in computing the corrected control is to establish that a state variation, $\delta \underline{s}_t$, exists. Methods for processing measurement data for this purpose are discussed in Chapter IV. Then the final state error is determined from (3-30).

$$\underline{\xi} = \Lambda_t^* \, \delta \underline{s}_t \tag{3-30}$$

The vector $\underline{\nu}^o$ is computed from the known quantities and the final state error using (3-77) through (3-82).

$$\underline{\nu}^o = M^{*-1} \underline{\xi} - M^{-1} \underline{\eta} \tag{3-84}$$

From (3-52) and (3-66) q^o and γ^o may be computed.

$$q^o = \left| B^T \Lambda^{*T} \underline{\nu}^o \right| \tag{3-52}$$

$$\left\{ \begin{array}{ll} \gamma^o = 0 & q^o < \dfrac{a_o}{2m} \\ \gamma^o = \dfrac{a_o}{mq^o} & q^o \geq \dfrac{a_o}{2m} \end{array} \right\} \tag{3-66}$$

Then from (3-68)

$$\underline{a}^o = -\gamma^o B^T \Lambda^{*T} \underline{\nu}^o \tag{3-68}$$

This result may be programmed into the vehicle control system to implement FTA guidance.

3.10 Discussion of the VTA Guidance Problem

In Chapter II it is shown that in field-free space, the minimum acceleration integral for VSI vehicles is $J = 6L^2/T^3$. A VTA guidance scheme which minimizes J for a given L in field-free space will therefore select an infinite transfer time unless constrained. If the planets were all in coplanar, circular orbits and the reference trajectory lay in this plane, a similar result would be obtained in the solar system. Because of the inclination and ellipticity of planetary orbits, local minima of J will occur which depend upon planet and spacecraft

orientation. A complete relaxation of the time constraint, t_f, in the theory of sections 3.8 and 3.9 would require that the new arrival time correspond at least to a local minimum of J. Therefore, unless t_f is approximately equal to this propellant-optimal arrival time, linear theory cannot be used to obtain propellant-optimal VTA guidance.

However, it is possible to restrict the change in arrival time, use linear theory and obtain useful solutions. A case where this is important occurs for a vehicle "in extremis". That is, insufficient power (or thrust) is available to null $\underline{\xi}$ at time t_f. The result obtained is dependent upon the method of restricting Δt and upon any simplifying assumptions. Consider the following example which is based upon the assumptions:

1) The target point relative to the planet center is unchanged.

2) The final velocity relative to the target point is unchanged.

3) The thrust acceleration $\underline{a}_f = \underline{a}(t_f)$ is constant over the interval.

The motion and position of the target point at the time $t_f + \Delta t$ are

$$\underline{r}_T(t_f + \Delta t) = \underline{r}_T(t_f) + \underline{v}_p \Delta t \qquad (3\text{-}85)$$

$$\underline{v}_T(t_f + \Delta t) = \underline{v}_T(t_f) = \underline{v}_p + \underline{v}_R + \underline{g}_T \Delta t \qquad (3\text{-}86)$$

where $\underline{v}_p$ and $\underline{v}_R$ are the planetary motion and the desired relative velocity, respectively, and $\underline{g}_T$ is the solar gravity at the planetary radius.

For the mission to be accomplished, the vehicle position and velocity must equal $\underline{r}_T$ and $\underline{v}_T$ at time $t_f + \Delta t$. The vehicle position and velocity in terms of the reference trajectory are

$$\underline{r}(t_f + \Delta t) = \underline{r}(t_f) + \delta\underline{r}(t_f) + \underline{v}(t_f)\Delta t + \underline{a}_f \left(\frac{\Delta t}{2}\right)^2 + \delta\underline{v}(t_f)\Delta t \qquad (3\text{-}87)$$

$$\underline{v}(t_f + \Delta t) = \underline{v}(t_f) + \delta\underline{v}(t_f) + \underline{a}_f(\Delta t) + \underline{g}_T \Delta t \qquad (3\text{-}88)$$

Solving equations (3-85) through (3-88) for $\delta\underline{r}_f$ and $\delta\underline{v}_f$ one obtains

$$\delta \underline{r}_f = - \underline{v}_R \, \Delta t - \delta \underline{v}_f \frac{\Delta t}{2} \tag{3-89}$$

$$\delta \underline{v}_f = - \underline{a}_f \, \Delta t \tag{3-90}$$

For the linear approximation assume $\delta v_f/2 \ll v_R$. Equations (3-89) and (3-90) may be inserted in the guidance equation.

$$-\begin{Bmatrix} \underline{v}_R \\ \underline{a}_f \end{Bmatrix} \Delta t = \underline{\xi} + \int_t^{t_f} \Lambda^* \, B \, \delta \underline{a} \, dt \tag{3-91}$$

Define a new miss vector $\underline{\xi}^*$ such that

$$\underline{\xi}^* = \begin{Bmatrix} \underline{\xi}^*_r \\ \underline{\xi}^*_v \end{Bmatrix} = \begin{Bmatrix} \underline{\xi}_r \\ \underline{\xi}_v \end{Bmatrix} + \begin{Bmatrix} \underline{v}_r \\ \underline{a}_f \end{Bmatrix} \Delta t \tag{3-92}$$

Then by selecting Δt such that the uncorrected final position error, $|\underline{\xi}^*_r|$, is a minimum, the vehicle will attain the terminus but at the time $t_f + \Delta t$ and with the least departure from the actual trajectory. The result is

$$\Delta t = \frac{-\underline{v}_R^T \, \underline{\xi}_r}{v_R^2} \tag{3-93}$$

Equation (3-93) may be inserted into (3-92) and the optimal control program found using the FTA procedures of sections 3.8 and 3.9.

If the target point is the planetary sphere of influence and the desired relative velocity is zero, obviously the above solution is invalid. Other assumptions may be used to treat such cases. The assumptions to be used and the criterion for restricting Δt may be changed to suit the purposes of the investigator. In general, the propellant-optimal VTA problem is not readily treated by linear methods unless additional constraints are used. Exploring the numerous possibilities that arise will be left for future investigations.

CHAPTER IV

ESTIMATE OF THE STATE VECTOR

4.1 Introduction

In this chapter a method of determining a "best estimate" of the vehicle state vector is presented. The term "best estimate" is used to designate the estimate for the state vector computed by processing redundant measurement data through a statistically optimum filter. Potter and Stern[32] have shown that all of the three commonly used methods of processing redundant data, namely, maximum likelihood, minimum error ellipsoid, and minimizing a characteristic scalar parameter all result in the same filter. They have shown further that for each unbiased optimum filter, there exists an associated biased optimum filter which produces a smaller error ellipsoid for the estimate.

Since all of the methods result in the same filter, in this thesis the method which presents the least mathematical complexity is used. The method involves a slight variation of the minimum error ellipsoid technique.

4.2 General Remarks

To establish a corrective thrust program it is desirable to have accurate knowledge of the state vector at the time the program is to be initiated. It is clear that given perfect knowledge of some prior state vector and of vehicle performance the prediction problem reduces simply to application of the state transition equation between the time for which the state is known and the time for which the prediction is desired. It is equally clear that perfect knowledge of a process seldom exists. Thus the problem becomes that of using existing information

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An unbiased filter will produce a true value for the state vector if all measurements are free of error. A biased filter is biased in favor of the a priori or prelaunch expectation of the second moment of the state vector probability density.

in such a way that its inaccuracies have minimum effect on the guidance decisions.

For ballistic vehicles the determination of position at two points along the trajectory suffices to specify the entire trajectory. Several investigators including Battin[30] and Stern[31], have studied the effects of using additional fixes and of selecting the geometry of individual celestial measurements to reduce the error in the trajectory determination. The techniques developed in those studies may be extended, in some cases without change, to the present work. The primary difference between the ballistic and continuous-thrust vehicles, is that the future portion of the actual continuous-thrust trajectory can never be completely determined on the basis of its history alone. This fact is due to the possible occurrence of random changes in the thrust vector. Another difference, which is more easily treated, is the first order dependence of the trajectory upon vehicle mass. This dependence may be handled by measuring propellant state at discrete intervals as well as making celestial measurements.

Thus for the continuous-thrust spacecraft, state determination from on-board measurements may be separated into two related problems:

1. determining the state history by suitable filtering of all measurements that have been made, and
2. predicting future values of the state vector.

The second of these is dependent upon but is not uniquely determined by the first.

The use of celestial sightings, as in the case of ballistic transfer, is sufficient to determine the spacecraft trajectory. However such measurements are not sufficient to specify the entire state history nor the variational history of the control vector which contributed to the state change. Because of this, prediction of the state vector solely from periodic celestial sightings and propellant measurement will be subject to larger uncertainties than if the prediction includes measurements of the control vector as well.

4.3 Unbiased Estimate of Present State

The term "present" state will be used to denote the state vector computed without regard to a computational time lag, thus, it is the state at the time of the most recent measurements.

The measurements which will be processed for the estimate are 1) periodic determination of position, 2) propellant state, and 3) continuous measurement of engine performance. It appears quite feasible to derive a method of including raw celestial observations in the computation process without explicitly deriving the position vector. However, this would contribute nothing to the present argument and will be left as a subject for future study. Consequently, the computed components of position variation will be considered as a "measurement". Since velocity is impractical to measure directly, except near planets, the velocity will be derived from successive position measurements.

The time of the present state estimate will be chosen as occuring between celestial "fixes". If the state at the time of a fix is desired, such an estimate will be a special case of the more general problem. This approach is justified on the basis that celestial observations may be separated by several days but current engine data is always available.

In the following development measured quantities will be denoted by the "tilde" (~) and the estimated quantity by a carat (∧).

The relation between the measurement quantities and the state may be written as:

$$N \, \delta \underline{s} = \delta \underline{q} \tag{4-1}$$

where N is a k by seven deterministic matrix relating $\delta \underline{s}$ to $\delta \underline{q}$.

$\delta \underline{s}$ is the state to be computed

$\delta \underline{q}$ is a column vector whose k components are the measurement data.

The matrix N may be derived directly from the state transition equation written between the time for which the estimate is desired, $t = t_n$ and any previous time, $t = t_i$.

$$\delta \underline{s}_i = T_{in}\, \delta \underline{s}_n - \Lambda_i^{-1} \int_{t_i}^{t_n} \Lambda\, B\, \delta \underline{a}\, dt \qquad (4\text{-}2)$$

where $T_{in} = \Lambda_i^{-1} \Lambda_n$. For simplicity, the integral term in equation (4-2) may be considered as a vector.

$$\Lambda_i^{-1} \int_{t_i}^{t_n} \Lambda B\, \delta \underline{a}\, dt = \begin{Bmatrix} \Delta \underline{r} \\ \Delta \underline{v} \\ \Delta m \end{Bmatrix}_i \qquad (4\text{-}3)$$

and the transition matrix T_{in} as a partitioned matrix of partial derivatives. Thus, if only the position vector at t_i is considered, one obtains

$$\delta \underline{r}_i + \Delta \underline{r}_i = \begin{bmatrix} \dfrac{\partial \underline{r}_i}{\partial \underline{r}_n} & \dfrac{\partial \underline{r}_i}{\partial \underline{v}_n} & \dfrac{\partial \underline{r}_i}{\partial m} \end{bmatrix} \begin{Bmatrix} \delta \underline{r} \\ \delta \underline{v} \\ \delta m \end{Bmatrix}_n \qquad (4\text{-}4)$$

where the matrix of partial derivatives consists of the first three rows of T_{in}. Equation (4-4) may be written for any number of times t_i $(i = n - 1, n - 2 \ldots .)$. An analogous equation may be written for the propellant measurement and the results arranged in the form of equation (4-1). One might expect the propellant measurement to be less critical and less subject to error than the other measurements. Consequently it may be necessary to include only a few measurements of propellant, perhaps one or two.

Let us consider now the problem of finding an unbiased filter, F_O, which minimizes the error in the state vector and satisfies the relation

$$\delta \hat{\underline{s}} = F_O\, \delta \tilde{\underline{q}} \qquad (4\text{-}5)$$

If the uncertainty in the state vector is $\underline{u}$ and the error vector associated with measurement is $\underline{\epsilon}$, then

$$\delta \tilde{\underline{q}} = \delta \underline{q} + \underline{\epsilon} = N\, \delta \underline{s} + N \underline{u} \qquad (4\text{-}6)$$

Since the true measurements and the actual state vector satisfy equation (4-1), then

$$\underline{\epsilon} = N \underline{u} \qquad (4\text{-}7)$$

The equation for the error ellipsoid associated with the measurement vector is given by

$$\underline{\epsilon}^T E^{-1} \underline{\epsilon} = 1 \tag{4-8}$$

where

$$E = \langle \underline{\epsilon}\ \underline{\epsilon}^T \rangle \tag{4-9}$$

is a k by k matrix and the brackets denote the mean value. Equations (4-6), (4-7), and (4-8) may be combined to yield

$$\underline{u}^T N^T E^{-1} N\underline{u} = (\delta \underline{\tilde{q}}^T - \delta \underline{s}^T N^T) E^{-1} (\delta \underline{\tilde{q}} - N\delta \underline{s}) \tag{4-10}$$

If the partial derivative of equation (4-10) with respect to the components of the state vector are set equal to zero, one obtains:

$$2 \left(\frac{\partial \underline{u}^T}{\partial s_i} \right) N^T E^{-1} N\underline{u} = -2 N^T E^{-1} (\delta \underline{\tilde{q}} - N\delta \underline{s}) = 0 \tag{4-11}$$

The solution to equation (4-11) is the estimate, $\delta \underline{\hat{s}}$.

$$\delta \underline{\hat{s}} = (N^T E^{-1} N)^{-1} N^T E^{-1} \delta \underline{\tilde{q}} \tag{4-12}$$

Thus for the unbiased filter, equation (4-12) yields:

$$F_O = (N^T E^{-1} N)^{-1} N^T E^{-1} \tag{4-13}$$

It is not surprising that the filter of equation (4-13) is the same filter derived by Potter and Stern using the method of maximum likelihood.[32]

4.4 Biased Estimate of Present State

The proof that a biased estimate will result in a smaller ellipsoid of error will be omitted since this topic is well covered in the literature[30,31,32]. In this section only the method of obtaining the biased filter from the unbiased filter will be presented.

The optimum biased filter, F_{OB}, is obtained by deleting the last seven columns from an associated unbiased filter which is computed using a fictitious measurement of present state. Thus the measurement error vector to use for computation of the error covariance matrix E is

$$\underline{\epsilon} = \begin{Bmatrix} \underline{\epsilon}_{n-1} \\ \underline{\epsilon}_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ -\delta\underline{s} \end{Bmatrix} \tag{4-14}$$

The last seven rows of the N matrix must then be the seven by seven identity matrix. The resulting unbiased filter F_O computed from equation (4-13) but with the last seven columns deleted defines the optimum biased filter, F_{OB}.

4.5 Covariance Matrix of Measurement Error

The matrix E is extremely important in the computation of the state estimate. Since it involves the measurement errors from both discrete and continuous measurements, it is worthy of closer examination.

In section 4.3 a subvector of the measurement vector $\delta\tilde{\underline{q}}$ was written in the form

$$\delta\tilde{\underline{q}}_i = \delta\tilde{\underline{r}}_i + \Delta\tilde{\underline{r}}_i \tag{4-15a}$$

From equation (4-3) it is apparent that

$$\Delta\tilde{\underline{r}}_i = [\, I_3 \; O_3 \; \underline{O}_3] \; \Lambda_i^{-1} \int_i^n \Lambda B \; \delta\tilde{\underline{a}} \, dt \tag{4-15}$$

where I_3 is the three by three identity, O_3 is a three by three null matrix and $\underline{O}_3$ a three component null vector. Thus a measurement of position consists of two parts: the celestial fix, $\delta\tilde{\underline{r}}_i$, at time $t = t_i$, and an additional vector $\Delta\tilde{\underline{r}}_i$ which contains the integrated engine variations from t_i to t_n. As a consequence, the error vector at time t_i will also consist of two parts,

$$\underline{\epsilon}_i = \underline{\epsilon}_{ri} + [\, I_3 \; O_3 \; \underline{O}_3 \,] \; \Lambda_i^{-1} \int_{t_i}^{t_n} \Lambda B \, \underline{\epsilon}_a \, dt \tag{4-16}$$

which for simplicity will be written as

$$\underline{\epsilon}_i = \underline{\epsilon}_{ri} + \int_{t_i}^{t_n} D_i \underline{\epsilon}_a \, dt \qquad (4\text{-}17)$$

where $\underline{\epsilon}_{ri}$ is the error in determining position and $\underline{\epsilon}_a$ is the error in measuring the engine quantity.

It appears justifiable to assume that measurement errors of different types, i. e. position, propellant flow, acceleration, are independent and have zero means. Further, that the position measurement errors for two different times are independent. The assumption of a zero mean for measurement errors does not result in a loss of generality, however it greatly simplifies the mathematics.

With the above assumptions, the E matrix includes terms of the following form:

1) On the diagonal

$$< \underline{\epsilon}_{ri} \underline{\epsilon}_{ri}^T > + < \int_{t_i}^{t_n} D_i \underline{\epsilon}_a \, dt \int_{t_i}^{t_n} \underline{\epsilon}_a^T D_i^T \, dt > \qquad (4\text{-}18)$$

2) Off the diagonal

$$< \int_{t_i}^{t_n} D_i \underline{\epsilon}_a \, dt \int_{t_j}^{t_n} \underline{\epsilon}_a^T D_j^T \, dt > \qquad (4\text{-}19)$$

Using the theory for handling random processes[38] the integral terms are easily reduced to the form

$$\int_{t_i}^{t_n} D_i (t_1) \int_{t_j}^{t_n} \theta (t_1 t_2) D_j^T (t_2) \, dt_1 \, dt_2 \qquad (4\text{-}20)$$

where $\theta (t_1 t_2)$ is a diagonal matrix of autocorrelation functions. To proceed it is necessary to make some assumption about θ. The most easily justified assumption is that the engine measurement errors each contains much higher frequency components than does the matrix D_i and each is uncorrelated except over short intervals. Thus, over the period of integration the measurement errors approach a "white

noise" distribution. With this assumption θ is reduced to a diagonal matrix of constants representing the measurement error variance multiplied by the Dirac delta function.

$$\theta = \begin{bmatrix} \sigma^2_{\dot{m}} & 0 \\ 0 & \sigma^2_c \end{bmatrix} \delta (t_2 - t_1) \tag{4-21}$$

(Measurement of propellant flow and exhaust velocity are used as examples in (4-21)) Designate, for simplicity

$$\theta = \sum \delta (t_2 - t_1) \tag{4-22}$$

where $\sum$ is a diagonal matrix of measurement variance.

One integration of the terms in (4-18) and (4-19) may be performed to yield

$$\int_{t_k}^{t_n} D_i \sum D_j^T \, dt \tag{4-23}$$

where the interval k to n represents the shorter of the intervals i to n or j to n. The E matrix formed from these elements is a symmetric positive definite matrix.

The necessity for including engine measurements in the estimate of state for low-thrust vehicles is made evident by equation (4-24). Note that the measurement error vector, when position data alone is used, would contain integrals of the actual engine variations. That is

$$\underline{\epsilon}_i = \underline{\epsilon}_{ri} + \int_{t_i}^{t_n} D_i \, \delta \underline{a} \, dt \tag{4-24}$$

Assuming that the engine measurement errors are much smaller than the engine variations to be measured, equation (4-24) would result in a sizeable increase in the error ellipsoid. In addition, as the interval i to n increases, the value of the celestial fix at time t_i is rapidly degraded due to the effect of the increasing value of the integral term in the E matrix.

4.6 Estimate of Future State

As noted previously, the future state vector of a continuous-thrust vehicle is not a deterministic function of the state history due to short period random variations and, perhaps, long term degradation of engine performance. If engine degradation, presumably in the form of loss of specific impulse, does not become a factor, the space navigator would be justified in assuming a zero mean for engine variations. With such an assumption a filter, biased to include a priori information of engine statistics and a priori information on state statistics, may be constructed which will produce an optimum prediction of future state.

If long term variation in engine performance is evident, the navigator is faced with the problem of extrapolating accumulated engine data in order to make a prediction with acceptable confidence. Certainly, sophisticated analytic techniques exist for smoothing and extrapolating the measured engine data. However, in this section we shall only be concerned with using data, regardless of the manner in which they are processed, to predict a future state. For example, simple graphical extrapolation of plotted data will suffice.

Two prediction techniques will be presented, 1) a very simple extrapolation technique for short term prediction and 2) an optimum filter technique for long term prediction. The first method assumes the present state has been determined by the methods of section 4.3 or 4.4. It uses the state transition equation and an estimate of engine performance to predict state in the near future. The estimate is

$$\delta\hat{\underline{s}}_{n+1} = T_{n+1\,n}\,\delta\hat{\underline{s}}_n + \Lambda^{-1}_{n+1}\int_{t_n}^{t_{n+1}} \Lambda B\,\delta\hat{\underline{a}}\,dt \qquad (4\text{-}25)$$

where the notation $t = t_{n+1}$ will indicate the future. This method may be characterized as an extrapolation of filtered data and may be quickly computed.

The second method uses the optimum filter of section 4.3 but in this case a representative subvector of the measurement vector is

$$\delta \underline{\tilde{q}}_i = \delta \underline{\tilde{r}}_i + \int_{t_i}^{t_n} D_i \, \delta \underline{\tilde{a}} \, dt + \int_{t_n}^{t_{n+1}} D_i \, \delta \underline{\hat{a}} \, dt \tag{4-26}$$

the error vector associated with the i^{th} position measurement is

$$\underline{\epsilon}_i = \underline{\epsilon}_{ri} + \int_{t_i}^{t_n} D_i \underline{\epsilon}_a \, dt + \int_{t_n}^{t_{n+1}} D_i \underline{\hat{\epsilon}}_a \, dt \tag{4-27}$$

where $\delta \underline{\hat{a}}$ and $\underline{\hat{\epsilon}}_a$ represent the extrapolation of engine performance and the error in that extrapolation, respectively.

The optimum biased filter of section 4.4 may be used without alteration; however, the covariance matrix of measurement error will be more difficult to compute.

This second method may be characterized as a filtering of extrapolated data and should be used for long range prediction. The proof by Potter and Stern[32] shows this second method to be optimum and intuitively it appears to be of correct form. By comparing expressions for the covariance matrix of uncertainty in state at time t_{n+1} for the two methods.

$$U = \langle \underline{u} \, \underline{u}^T \rangle \tag{4-28}$$

it is apparent that both methods result in the same error ellipsoid as t_{n+1} approaches t_n. For both methods:

$$U \underset{t_{n+1} \to t_n}{=} (N^T E^{-1} N)^{-1} \tag{4-29}$$

thus the assertion that the simple technique will be quite adequate for short term predictions appears justified.

CHAPTER V

TRAJECTORY DETERMINATION

5.1 Summary of Chapter V

In this chapter the problem of finding a trajectory which satisfies the end condition, the propulsion restriction and which minimizes propellant consumption is discussed and illustrated with the VSI trajectory. The problem is formulated using the classical methods of the calculus of variations. A new interpretation is presented for the results, which are shown to yield the same control program as the Pontryagin maximum principle. The guidance theory of Chapter III is therefore suggested as a trajectory computation scheme. Finally, qualitative aspects of low-thrust optimal trajectories are discussed.

5.2 General Remarks

A vast amount of effort has been directed in recent years toward the study of optimization problems. Such problems belong to the calculus of variations which owes its early development to such men as Lagrange, Euler, Hamilton and Gauss. The introduction of high speed computers has been the primary impetus in bringing renewed interest, after years of limited application, to variational techniques. Specifically, much recent literature treats the characteristics of, the necessary conditions for, and methods of computing solutions to optimization problems. Variational techniques are used almost exclusively as the primary mathematical tool. The contributions listed as references in this report constitute only a minor fraction of the published works. These efforts have resulted in a large body of theory now called optimal control theory which has application to virtually all optimization problems dealing with dynamical systems.

A fundamental precept of the theory is that along optimal trajectories, admissible first order variations in an unconstrained control program cannot produce a first order effect in the cost function. For

unconstrained controls, admissible control variations are those which do not cause a first order change in boundary conditions. This principle is derived directly from basic equations and is part of the definition of optimality in the calculus of variations. A mathematical consequence of this principle is that certain arrays of coefficients, when evaluated along the optimal trajectory, will have a zero determinant. When working with the practical problem of computing trajectories, or of guidance around the optimal trajectory, the inverse of a matrix defined by such arrays usually appears in the equations. Obviously the occurrence of a singularity complicates the procedure of finding an optimal control program and its associated optimal trajectory. It has led to investigation of second variations of the cost and the trajectory[39], to various schemes for finding near-optimal controls[12], and to optimal controls which only approach the desired boundary conditions[14].

The implication of these investigations is disturbing from a physical viewpoint since they imply that an otherwise well behaved, smoothly operating system, in some sense becomes uncontrollable along an optimal reference trajectory. In reality, singularities are more often mathematical than physical. A further implication is, that although a control can be found which approaches the optimal control to within a small increment, it is orders of magnitude more difficult to find the exact optimal control.

The physical world does not usually behave in such an unruly manner, thus the answer must be: 1) the mathematics have an interpretation that has been overlooked or 2) the problems may be approached from a different viewpoint.

Actually, both 1) and 2) have validity. To support this contention, the problem of generating an optimal reference trajectory for use in testing the guidance theory of Chapter III is considered as an example. To be sure, a conservative field such as the gravitational field is a well behaved space in which to work. Undoubtedly, problems which deal with dissipative forces or higher order nonlinearities may present difficulties not readily resolved by the method of this chapter. Hopefully, however, it will provide an approach that can serve as a starting point for more difficult problems.

5.3 The Calculus of Variations Problem

In Chapter II it is shown that minimizing the acceleration integral is equivalent to minimizing the propellant consumption for a low-thrust vehicle. Thus the mass rate equation is superfluous for VSI trajectories and it is necessary only to work with the equations of motion and the acceleration integral. For the VSI vehicle the problem can be formulated as: Given

$$\underline{r}(0) = \underline{r}_o \tag{5-1}$$

$$\underline{v}(0) = \underline{v}_o \tag{5-2}$$

satisfy

$$\underline{r}(t_f) = \underline{r}_f \tag{5-3}$$

$$\underline{v}(t_f) = \underline{v}_f \tag{5-4}$$

subject to

$$\dot{\underline{r}} = \underline{v} \tag{5-5}$$

$$\dot{\underline{v}} = -\frac{u}{r^3}\underline{r} + \underline{a} \tag{5-6}$$

and minimize

$$J = \frac{1}{2}\int_0^{t_f} a^2\, dt \tag{5-7}$$

A scalar functional, F, will now be formed using the well known Lagrange multiplier technique.

$$F = \underline{\lambda}_r^T(\dot{\underline{r}} - \underline{v}) + \underline{\lambda}_v^T\left(\dot{\underline{v}} + \frac{u}{r^3}\underline{r} - \underline{a}\right) + \frac{a^2}{2} \tag{5-8}$$

where $\underline{\lambda}_r$ and $\underline{\lambda}_v$ are time varying Lagrange multiplier vectors (or Euler variables). The remaining variables have the same meanings as in previous chapters.

Since the bracketed terms are zero, clearly

$$J = \int_0^{t_f} F\, dt \tag{5-9}$$

Integrating by parts and setting the first variation of the integral equal to zero, one obtains

$$\delta \int_0^{t_f} F\, dt = 0 = \left[\underline{\lambda}_r^T\, \delta \underline{r} + \underline{\lambda}_v^T\, \delta \underline{v} \right]_0^{t_f}$$

$$- \int_0^{t_f} \left[(\dot{\underline{\lambda}}_r^T - \underline{\lambda}_v^T G)\, \delta \underline{r} + (\dot{\underline{\lambda}}_v^T + \underline{\lambda}_r^T)\, \delta \underline{v} \right. \tag{5-10}$$

$$\left. + (\underline{a}^T - \underline{\lambda}_v^T)\, \delta \underline{a} \right] dt$$

Applying the fixed end conditions and setting the coefficients of the state and control variations equal to zero gives the Euler equations and the boundary conditions.

$$\begin{Bmatrix} \dot{\underline{\lambda}}_r \\ \dot{\underline{\lambda}}_v \end{Bmatrix}^T = - \begin{Bmatrix} \underline{\lambda}_r \\ \underline{\lambda}_v \end{Bmatrix}^T \begin{bmatrix} O_3 & I_3 \\ -G & O_3 \end{bmatrix} \tag{5-11}$$

$$\begin{Bmatrix} \underline{\lambda}_r \\ \underline{\lambda}_v \end{Bmatrix}_{t_f} = ? \tag{5-12}$$

$$\begin{Bmatrix} \underline{\lambda}_r \\ \underline{\lambda}_v \end{Bmatrix}_{0} = ? \tag{5-13}$$

$$\underline{a} = \underline{\lambda}_v \tag{5-14}$$

Notice that the Euler variables, (5-11), satisfy the adjoint relationship to the variational equations of motion and that the boundary conditions are unknown. Equations (5-12) and (5-13) are written to emphasize the unknown boundary conditions.

Clearly, if the correct initial value of the six component Euler vector were known, the entire system of state and Euler equations could be integrated simultaneously from $t = 0$ to $t = t_f$. The unique reference trajectory and its associated optimal control program would then be known. A procedure for finding the initial value is discussed in later sections.

A very annoying fact about the preceding approach to finding extremals in the calculus of variations is the necessity of searching for a formulation of the problem which gives a meaningful result. Minimizing the cost function J is known to be equivalent to minimizing propellant consumption, thus one differential equation of state was eliminated, in particular, the differential equation describing the rate of change of the optimized quantity, mass. A simple answer resulted. It should not be necessary to search for an equivalent formulation for a problem if the set of differential equations describing the system is complete, linearly independent and reasonably well behaved. Two different Mayer formulations of the VSI problem help illustrate this argument.

For the first problem, minimize propellant consumption instead of the acceleration integral. Equations (5-1) through (5-6) hold, but the mass rate equation must be reintroduced. Thus it is required that

$$\dot{m} = - \frac{a^2 m^2}{2p} \tag{5-15}$$

and that

$$m_p = \int_0^{t_f} - \dot{m} \, dt = m_o - m_f \tag{5-16}$$

be a minimum where m_p is propellant mass. For fixed initial mass the first variation of m_p equals the variation of $- m_f$. Forming the functional F

$$F = \underline{\lambda}_r^T (\dot{\underline{r}} - \underline{v}) + \underline{\lambda}_v^T (\dot{\underline{v}} + \frac{u}{r^3} \underline{r} - \underline{a}) + \lambda_m (\dot{m} + \frac{a^2 m^2}{2p}) - \dot{m} \tag{5-17}$$

then integrating by parts and setting the first variation of the integral equal to zero as before, yields

$$\begin{aligned} \delta \int_0^{t_f} F \, dt = 0 = & \left[\underline{\lambda}_r^T \delta \underline{r} + \underline{\lambda}_v^T \delta \underline{v} + (\lambda_m - 1) \delta m \right]_0^{t_f} \\ & - \int_0^{t_f} \Big[(\dot{\underline{\lambda}}_r^T - \underline{\lambda}_v^T G) \delta \underline{r} \\ & + (\dot{\underline{\lambda}}_v^T + \underline{\lambda}_r^T) \delta \underline{v} + (\dot{\lambda}_m - \lambda_m \frac{a^2 m}{p}) \delta m \\ & + (\underline{\lambda}_v^T - \lambda_m \frac{m^2}{p} \underline{a}^T) \delta \underline{a} \Big] dt \end{aligned} \tag{5-18}$$

In this case the Euler variables for position and velocity again satisfy equations (5-11) through (5-13). In addition

$$(\lambda_m - 1)_0 = ? \tag{5-19}$$

$$\lambda_{m_f} = 1 \tag{5-20}$$

$$\dot{\lambda}_m - \lambda_m \frac{a^2 m}{p} = 0 \tag{5-21}$$

$$\underline{a} = \frac{\underline{\lambda}_v}{\lambda_m \dfrac{m^2}{p}} \tag{5-22}$$

Equation (5-22) appears to differ from (5-14) by a scalar function of time. Since the control program is unique (5-22) must reduce to (5-14). By manipulating (5-15) and (5-22) it is possible to obtain

$$\frac{\dot{\lambda}_m}{\lambda_m} = -2 \frac{\dot{m}}{m} \tag{5-23}$$

which has the solution

$$\lambda_m = \frac{\text{constant}}{m^2} \tag{5-24}$$

Applying the boundary condition (5-20)

$$\lambda_m = \frac{m_f^2}{m^2} \tag{5-25}$$

Substituting into (5-22)

$$\underline{a} = \frac{p}{m_f^2} \underline{\lambda}_v \tag{5-26}$$

Equation (5-26) differs from (5-14) only by a constant which is easily absorbed in the unknown boundary conditions on $\underline{\lambda}_v$. The Euler variable λ_m appears to have been superfluous.

Consider a slightly different formulation which is just as valid. Use thrust as the control then

$$\dot{\underline{r}} = \underline{v} \tag{5-5}$$

$$\dot{\underline{v}} = -\frac{u}{r^3}\underline{r} + \frac{\underline{f}}{m} \tag{5-27}$$

$$\dot{m} = -\frac{f^2}{2p} \tag{5-28}$$

minimize

$$m_p = \int_0^{t_f} -\dot{m}\,dt \tag{5-29}$$

Proceeding as before, the functional F and its first variation are

$$F = \underline{\lambda}_r^T(\dot{\underline{r}} - \underline{v}) + \underline{\lambda}_v^T(\dot{\underline{v}} + \frac{u}{r^3}\underline{r} - \frac{\underline{f}}{m}) + \lambda_m(\dot{m} + \frac{f^2}{2p}) - \dot{m} \tag{5-30}$$

$$\begin{aligned} \delta\int_0^{t_f} F\,dt = 0 = \Big[\qquad\qquad \Big]_0^{t_f} - \int_0^{t_f}\Big[\Big(\qquad\qquad \Big)\delta\underline{r} \\ + \Big(\qquad \Big)\delta\underline{v} + (\dot{\lambda}_m - \underline{\lambda}_v^T\frac{\underline{f}}{m^2})\,\delta m \\ + \Big(\frac{\underline{\lambda}_v^T}{m} - \frac{\lambda_m}{p}\underline{f}^T\Big)\,\delta\underline{f}\Big]\,dt \end{aligned} \tag{5-31}$$

The blank brackets contain the same terms as (5-18) and are used to show the differences between the two formulations. In this case

$$\dot{\lambda}_m = \underline{\lambda}_v^T\frac{\underline{f}}{m^2} \tag{5-32}$$

$$\underline{f} = \frac{\underline{\lambda}_v}{\lambda_m\frac{m}{p}} \tag{5-33}$$

Using (5-33) to eliminate $\underline{\lambda}_v^T$ from (5-32) and using the state equation (5-28), it is found that again

$$\dot{\lambda}_m = -2\lambda_m \frac{\dot{m}}{m} \qquad (5\text{-}23)$$

Solution (5-25) holds thus

$$\underline{f} = \frac{p}{m_f^2} m \underline{\lambda}_v \qquad (5\text{-}34)$$

which is the same as (5-26) since $\underline{a} = \underline{f}/m$.

The preceding VSI examples result in the same control and in none of them is it necessary to use the variable λ_m. Before interpreting what this means one additional example is presented which involves placing a constraint on the thrust. Assume that

$$f^2 \leq f_o^2 \qquad (5\text{-}35)$$

where f_o is the thrust limit. Following Breakwell[17], the constraint may be handled by adding to the functional (5-30) the term

$$\gamma' (f^2 - f_o^2) = 0 \qquad (5\text{-}36)$$

where γ' is selected to satisfy (5-36). Thus for this case

$$F = \left[(5\text{-}30)\right] + \gamma' (f^2 - f_o^2) \qquad (5\text{-}37)$$

$$\delta \int_0^{t_f} F\, dt = 0 = \Big[\qquad \Big]_0^{t_f} - \int_0^{t_f} \Big[(\qquad) \delta \underline{r} + (\qquad) \delta \underline{v}$$

$$+ \left(\dot{\lambda}_m - \underline{\lambda}_v^T \frac{\underline{f}}{m^2} \right) \delta m + \left(\frac{\underline{\lambda}_v^T}{m} - \lambda_m \frac{\underline{f}^T}{p} - \gamma' \underline{f}^T \right) \Big] \delta \underline{f}$$

$$(5\text{-}38)$$

where the blank brackets are the same as (5-31) and not needed in the argument.

Consider the coefficients of δm and $\delta \underline{f}$ in (5-38)

$$\dot{\lambda}_m - \underline{\lambda}_v^T \frac{\underline{f}}{m^2} = 0 \qquad (5\text{-}39)$$

$$\underline{f} = \frac{\underline{\lambda}_v}{m\left(\gamma' + \frac{\lambda_m}{p}\right)} \qquad (5\text{-}40)$$

Eliminating $\underline{\lambda}_v^T$ in (5-39) by using (5-40)

$$\dot{\lambda}_m = \left(\gamma' + \frac{\lambda_m}{p}\right)\frac{f^2}{m} \qquad (5\text{-}41)$$

When $f \neq f_o$, γ' must be zero to satisfy (5-36). Therefore the solutions for λ_m and $\underline{f}$ in such unconstrained regions can differ from (5-25) and (5-34) at most by a constant. When $f = f_o$ the only admissible solution for (5-40) is

$$\underline{f} = \frac{f_o \underline{\lambda}_v}{|\underline{\lambda}_v|} \qquad (5\text{-}42)$$

Therefore

$$\left(\gamma' + \frac{\lambda_m}{p}\right) m = \frac{|\underline{\lambda}_v|}{f_o} \qquad (5\text{-}43)$$

With this result it is possible to rewrite (5-40) with a new variable γ such that

$$\underline{f} = m\gamma\,\underline{\lambda}_v \qquad (5\text{-}44)$$

where

$$\left\{\begin{array}{ll} \gamma = 1 & m\,|\underline{\lambda}_v| \leq f_o \\ \gamma = \dfrac{f_o}{m\,|\underline{\lambda}_v|} & m\,|\underline{\lambda}_v| > f_o \end{array}\right\} \qquad (5\text{-}45)$$

The Euler variable λ_m and its differential equation have again been eliminated from the problem.

In the preceding examples the manipulations required to remove λ_m are not particularly difficult but by no means are they obvious. The meaning of a superfluous Euler variable immediately comes into question and it seems reasonable to inquire if there is general significance

in this phenomenon or if the preceding examples are only isolated special cases. Conceptually there is nothing about the variable λ_m which is unique except for its association with the cost variable m in a Mayer form problem. In reference 17 Breakwell interprets the Euler variables as the sensitivity of the absolute minimum of the cost function (i. e. the unconstrained minimum) to changes in the state variables. Therefore for the problem of this thesis

$$\underline{\lambda}^T(t) = \frac{\partial m_p^*}{\partial \underline{s}_t} = - \frac{\partial m_f^*}{\partial \underline{s}_t} \tag{5-46}$$

where m_p^* denotes the minimum value of m_p obtainable from the initial state when no control constraints are used; similarly for $- m_f^*$.

In a typical two point boundary value problem with physical rates of the form $\dot{\underline{s}} = \underline{g}\,(\underline{s}, \underline{a})$, the usual objective is to find a control solution which satisfies the end point. If any of the state variables have free end conditions the solution is not unique. Optimization criterion are then used to assure uniqueness. From Breakwell's interpretation, when the Euler variables are formed into a vector they describe the direction in state space of maximum sensitivity of the cost, i. e. the gradient. However for state variables involved in the cost function such information is available directly from the governing differential equations. In answering the question: "How do state variations affect the cost?", the Euler variables provide a coupling between state variables and the cost function which for some variables is not evident from the system equations. But for state variables involved in the cost this coupling is essentially a priori information and is in some sense superfluous.

The classical approach in the calculus of variations assigns an Euler variable for each state variable. If a variable is superfluous it should drop out of the formulation. But finding a way to assure that it does may be exceedingly difficult, even when it is recognized that the variable is superfluous. As is subsequently shown, the Euler variables correspond to the costate variables in Pontryagin formulations. If the variable is superfluous in the calculus of variations it is also superfluous

using the Hamiltonian approach. This fact was tacitly acknowledged in Chapter III when the state variation, δm_f, was dropped from consideration and the last row of the adjoint set was deleted. The reason given in Chapter III for deleting one row of adjoint functions was different from here, but the result is the same.

The general significance of a superfluous Euler variable is indicated by comparing the classical approach to the newer state space approach using the Pontryagin maximum principle. The classical formulation suffers from several difficulties, two of which are: 1) The number of variables and equations are usually quite large and, perhaps due to tradition, are often treated individually as scalars. As a result algebraic detail often obscures important ideas in classical formulations. The more compact vector and matrix notation are only now becoming widespread in the literature. 2) The theory offers no suggestion of how to solve for the correct boundary conditions on the Euler variables.

However, an important attribute of the more detailed notation is indicated by the preceding examples. By sufficient manipulation an unneeded variable was eliminated. It is more than coincidence that the variable is precisely the one that creates problems in the state space approach.

The state space approach using vector and matrix notation as in Chapter III is popular because of its compactness. Further, when coupled with the Pontryagin maximum principle it may be used to solve optimization problems including cases of linear control (the CSI problem) which can not be completely solved in the calculus of variations. The approach also provides a method of determining the boundary values for the costate variables. Finally, the optimality criterion is more useful. This last statement is discussed fully in Appendix D.

In the state space approach the system differential equations are written in vector form, an appropriate cost function is chosen and then methods are sought to find the desired solution. From the references previously mentioned it is apparent that the solutions often encounter a singular matrix. In the problem of this thesis the singular matrix

is denoted as M_o and is the seven by seven matrix corresponding to M in Chapter III. These matrices are discussed in detail in Appendix C. It is suggested that such singular matrices may be the result of attempting to optimize a function of state and at the same time retain a costate variable (or Euler variable) which is superfluous. In the problem of this thesis such was the case. It is further suggested that difficulties sometimes encountered in numerically finding terminal values for the Euler variables may also be due to the presence of a superfluous Euler variable which was not recognized as such.

No attempt will be made in this report to derive the general rule which will show when an Euler variable is superfluous and thus may be eliminated a priori. It may be the case that those matrix elements which cause nonphysical singularities to occur in state space formulations can always be removed without changing the problem.

In the remainder of this chapter the relationship of the calculus of variations solution to the state space solution is shown for the low-thrust problem only. The method of deleting λ_m is illustrated.

5.4 Removal of a Superfluous Euler Variable

Although an Euler variable may be superfluous, a general method of removing one is not obvious from the derivation in section 5.3. Consider, however, the following derivation of the problem in section 5.3 which combines the compact notation with the classical approach. It may be indicative of a general method.

Assume the system is described by physical rates which have the form

$$\dot{\underline{s}} = \underline{g}\,(\underline{s},\, \underline{a}) \tag{5-47}$$

where $\underline{s}$ is the state vector, $\underline{a}$ the control vector and $\underline{g}$ is a vector function of state and control.

Assume that the cost variable is some scalar function of the physical rates or some nonlinear function of the control.

$$\dot{S} = h\,(\dot{\underline{s}}) \tag{5-48}$$

or

$$S = h\,(\underline{a}) \qquad (5\text{-}49)$$

where $\dot{S}$ is the cost integrand.

Further, allow the physical rates to be subjected to constraints. Define the constrained variables as a vector function of the physical rates or of the control. That is

$$\underline{y} = \underline{y}\,[\underline{g}\,(\underline{s}, \underline{a})] \qquad (5\text{-}50)$$

A general functional, equivalent to those used in section 5.3 is

$$F = \dot{S} + \underline{\lambda}^T \underline{g} + \underline{\gamma}'^{\,T} \underline{y} \qquad (5\text{-}51)$$

where $\underline{\lambda}$ is the vector of Euler variables and where $\underline{\gamma}'$ is determined in such a fashion that the constraint is satisfied. It is a vector equivalent of the γ' in section 5.3.

Application of the usual variational technique produces the expanded set of state equations, Euler equations and control equations.

$$\dot{\underline{s}} = \underline{g}\,(\underline{s}, \underline{a}) \qquad (5\text{-}47)$$

$$-\dot{\underline{\lambda}} = \left(\frac{\partial \dot{S}}{\partial \underline{s}} + \underline{\lambda}^T \frac{\partial \underline{g}}{\partial \underline{s}} + \gamma'^{\,T} \frac{\partial \underline{y}}{\partial \underline{s}} \right)^T \qquad (5\text{-}52)$$

$$\underline{O} = \left(\frac{\partial \dot{S}}{\partial \underline{a}} + \underline{\lambda}^T \frac{\partial \underline{g}}{\partial \underline{a}} + \gamma'^{\,T} \frac{\partial \underline{y}}{\partial \underline{a}} \right)^T \qquad (5\text{-}53)$$

For the unrestricted case such that $\underline{\gamma}' \equiv \underline{O}$ and using the familiar definition of the matrices A and B, that is

$$\frac{\partial \underline{g}}{\partial \underline{s}} = A \qquad (5\text{-}54)$$

$$\frac{\partial \underline{g}}{\partial \underline{a}} = B \qquad (5\text{-}55)$$

then from (5-52) one obtains

$$\underline{\lambda}^T = -\underline{\lambda}^T A \qquad (5\text{-}56)$$

A solution to the set (5-53) to within an unknown scale factor, is

$$\underline{a} = B^T \underline{\lambda} \tag{5-57}$$

That (5-57) is a solution may be verified by expansion. The variational equations of state and their adjoint set are

$$\delta\dot{\underline{s}} = A\,\delta\underline{s} + B\,\delta\underline{a} \tag{5-58}$$

$$\dot{\Lambda} = -\Lambda A \tag{5-59}$$

Since the Euler equations (5-56) satisfy the adjoint relation as does (5-59) a solution for (5-56) is obtained by choosing

$$\Lambda_f = I \tag{5-60}$$

Then

$$\underline{\lambda}_t = \Lambda_t^T \underline{\lambda}_f \tag{5-61}$$

Dropping the subscript t from (5-61) and substituting into (5-57) yields

$$\underline{a} = B^T \Lambda^T \underline{\lambda}_f \tag{5-62}$$

By partitioning Λ and $\underline{\lambda}_f$ it is apparent that deleting the seventh row of Λ, $\underline{\Lambda}_7^T$, and deleting the unknown boundary value λ_{m_f} eliminates the superfluous Euler variable from consideration. That is

$$\underline{a} = B^T \left[\Lambda^{*T} \;\middle|\; \underline{\Lambda}_7 \right] \left\{ \begin{array}{c} \underline{\lambda}_r \\ \underline{\lambda}_v \\ \hdashline \lambda_m \end{array} \right\}_f \tag{5-63}$$

In order to find the optimal trajectory it is only necessary to find the final value of the six Euler variables. (Or, since the adjoint and fundamental solutions, discussed in Chapter III, allow transformation at will between one terminus and the other; the initial values of the Euler variables may be used.)

By letting

$$\underline{\nu} = \left\{ \begin{array}{c} \underline{\lambda}_r \\ \underline{\lambda}_v \end{array} \right\}_f \tag{5-64}$$

and deleting $\underline{\Lambda}_7$ and λ_{m_f} from (5-63), the expression yields

$$\underline{a} = B^T \Lambda^{*T} \underline{\nu} \tag{5-65}$$

which is identical with the expression for the optimal control program for guidance derived in Chapter III. Thus both the Pontryagin principle and the Euler variables give the same result when λ_m is deleted. It is shown in Appendix C that along the optimal trajectory $\underline{\lambda}_f$ is a null vector of $B^T \Lambda^T$, thus equation (5-62) displays the well known and very troublesome singularity when evaluated along the optimal trajectory. This is the consequence of optimality which has instigated the search for new methods, including this one. It is further shown that $\underline{\nu}$ is not a null vector of $B^T \Lambda^{*T}$ thus (5-65) can be used.

For the restricted VSI case the preceding arguments hold with γ' entering as in section 5.3. The restricted VSI case is completely determinate in the calculus of variations using Breakwell's approach and deleting λ_m. The CSI case may also be treated in an analogous fashion except that the coast phase can not be uniquely determined. This is due to the so-called degeneracy of the calculus of variations for linear optimums[7].

5.5 Solution by Direct Integration

There are two general approaches for finding an optimal trajectory. Descriptions of the first method are usually prefaced by: "If any non optimal solution can be found which satisfies the boundary conditions, then the solution can be moved in the direction of the optimal, etc." This approach is usually called a gradient method. It is not used in this report because nonoptimal solutions cannot satisfy the Euler equations, thus it would be difficult to relate any nonoptimal control to the form of equation (5-65).

The second general approach is often described as "solving the wrong problem in an optimal manner." This is the approach used in Chapter VI. An initial guess is made for the initial Euler variables $\underline{\nu}$, then the state equations, adjoint set and Euler variables are integrated to the final time. The resulting final values are compared with the desired final values and a new estimate of $\underline{\nu}$ is computed. The process

is repeated until the boundary values are satisfied. The fact that the gravitational field is conservative and well behaved aids materially in improving the speed of convergence.

5.6 Properties of Optimal Trajectories

It is possible to gain considerable insight into the properties of propellant-optimal trajectories without resorting to machine computation. Such insight aids materially in working with the mathematics of the transfer process and often leads to new methods of approach.

For this purpose, consider the problem of transferring between two planets with minimum propellant expenditure as one of changing energy and angular momentum in the most efficient manner.

From the energy integral of orbital mechanics one may write the energy per unit mass of the vehicle as

$$e = \frac{v^2}{2} - \frac{\mu}{r} \tag{5-67}$$

and the energy time derivative as

$$\dot{e} = v\dot{v} + \frac{\mu}{r^2}\dot{r} \tag{5-68}$$

Equation (5-68) may be written in the vector notation and reduced with help of the equations of motion to

$$\dot{e} = \underline{v}^T \underline{a} \tag{5-69}$$

Equation (5-69) represents the rate of energy change imparted to the vehicle. The rate of energy expenditure by the propulsion system is the power

$$p = -\frac{\dot{m}c^2}{2} = -\frac{\dot{m}\,\underline{c}^T\underline{c}}{2} \tag{5-70}$$

where $\dot{m}$ is considered as the mass flow rate per unit mass and $\underline{c}$ has the opposite sense of $\underline{a}$.

Similarly a functional expression for angular momentum is

$$\underline{h} = \underline{r} \times \underline{v} \tag{5-71}$$

$$\dot{\underline{h}} = \dot{\underline{r}} \times \underline{v} + \underline{r} \times \dot{\underline{v}} \tag{5-72}$$

where the notation signifies the vector product of $\underline{r}$ and $\underline{v}$. Equation (5-72) is reduced with help of the equations of motions to

$$\dot{\underline{h}} = \underline{r} \times \underline{a} \tag{5-73}$$

Using equations (5-69) and (5-73) the problem is formulated as:

Minimize $$S = \int_0^{t_f} -\dot{m}\, dt \tag{5-74}$$

subject to the constraints

$$\Delta e = \int_0^{t_f} \underline{v}^T \underline{a}\, dt \tag{5-75}$$

$$\Delta \underline{h} = \int_0^{t_f} \underline{r} \times \underline{a}\, dt \tag{5-76}$$

plus velocity and position constraints and where Δe and $\Delta \underline{h}$ are known quantities.

The above formulation is not computationally desirable; however it provides a basis for some significant qualitative arguments.

It is apparent from equation (5-69) that with $\underline{a}$ fixed in magnitude the vehicle changes energy most rapidly, thus most efficiently, when v is large and $\underline{v}$ and $\underline{a}$ are colinear. From the energy equation (5-67) one observes that in a central force field the velocity magnitude goes inversely with the square root of the radius. From equation (5-73) it is apparent that $\underline{h}$ changes most rapidly when r is large and orthogonal to $\underline{a}$.

From the preceding observations one may deduce that the trajectory which minimizes the propellant expenditure will tend to keep the exhaust velocity vector aligned with the large velocity vector when the vehicle is deep in a gravitational field so that energy is changed most efficiently. It will also tend to rotate the angular momentum vector when the vehicle is far out in the gravitational field so that the vector $\underline{r}$ has large magnitude. Where these requirements are incompatable

with boundary conditions, the optimum solution should resolve the conflict in favor of propellant conservation.

The trajectories which have been generated numerically in this thesis all appear to satisfy these precepts in so far as boundary conditions permit. The increase in specific impulse which is characteristic of variable thrust rockets as they traverse the center portion of the heliocentric phase is due to the acceleration vector becoming orthogonal to the velocity vector. Likewise the coast phase for a thrust-limited rocket occurs when the acceleration vector rotates through the orthogonal orientation.

With the qualitative arguments of this section, it is possible to sketch a reasonable approximation to a low-thrust optimum trajectory without resorting to machine computation.

CHAPTER VI

COMPUTATIONAL PROCEDURE FOR TRAJECTORIES

6.1 Summary of Chapter VI

Details of a procedure for using the guidance equation as the basis for trajectory computations are set forth in this chapter. Specifically, the differential equations to be integrated and the procedure for correcting initial conditions on the Euler equations are presented.

6.2 General Remarks

In Chapter V the fundamental problem of selecting initial (or final) conditions on the Euler variables is introduced. Since the acceleration program is an explicit function of the Euler variables it is necessary to make a first estimate of the Euler initial conditions in order to start the iteration. This estimate will usually be a gross error under the best of circumstances, thus it is desirable to choose a very simple estimate such as the null vector or a unit vector. Because of this, the procedure we seek must be strongly convergent and independent of the error in the first iteration attempt.

In any iterative procedure the speed of convergence is directly dependent upon the validity of assuming that certain quantities do not change from one iteration to the next. In the preceding chapters it was explicitly assumed that the miss vector at the terminal point was dependent only upon a change in the initial Euler vector and that Λ and B were invariant for neighboring trajectories. For small perturbations around a reference trajectory such an assumption is valid. However for the large perturbations expected in trajectory computation Λ and B are not invariant. The change of these quantities from one iteration to the next and the change of other second order quantities will determine the speed of convergence of the procedure.

6.3 The Correction Procedure

From Chapter III one may write the optimal acceleration program as a scalar times a matrix function of the adjoint set and the initialization vector.

$$\underline{a}^o = \gamma^o \, B^T \, \Lambda^{*T} \, \underline{\nu}^o \tag{6-1}$$

where γ is the scalar switching function derived from the limiting condition on acceleration, B is the seven by three matrix of partial derivatives, $\partial \underline{g}/\partial \underline{a}$, Λ^* is the six by seven reduced adjoint set and $\underline{\nu}$ is the six by one initialization vector.

In order to simplify the notation in certain equations to follow, Euler variables will be used interchangeably with the matrix notation. Recall from Chapter V that

$$\underline{\lambda}_v = B^T \, \Lambda^{*T} \, \underline{\nu} \tag{6-2}$$

and

$$\dot{\underline{\lambda}}_v = - \underline{\lambda}_r \tag{6-3}$$

The objective of the iterative search for the optimal acceleration program is to find the vector $\underline{\nu}^o$ which causes the six component miss vector, $\underline{\xi}$, to approach the null vector to some desired accuracy. Thus

$$\underline{\nu}_n = \underline{\nu}_{n-1} + \Delta \underline{\nu} \tag{6-4}$$

and we desire that

$$\underline{\nu}_n = \underline{\nu}^o \tag{6-5}$$

for n as small as possible, where n is the iteration number.

The miss vector, $\underline{\xi}$, may be written as the difference in final state which results when a nonoptimal acceleration program, $\underline{a}$, is used in place of $\underline{a}^o$. Define

$$\underline{\eta} = \int_0^{t_f} \Lambda^* B \, \underline{a} \, dt = M \underline{\nu} \tag{6-6}$$

and

$$- \underline{\xi} = \int_0^{t_f} \Lambda^* \, B \, (\underline{a}^o - \underline{a}) \, dt \tag{6-8}$$

then using (6-1) and (6-8)

$$-\underline{\xi} = \int_0^{t_f} \Lambda^* \, B \, B^T \Lambda^{*T} \Delta(\gamma \, \underline{\nu}) \, dt \qquad (6\text{-}9)$$

Equation (6-9) implicitly assumes that Λ and B are invariant functions of time and thus constant from one iteration to the next. The assumption permits adequate convergence and is used throughout.

In order to provide for the discontinuities which occur in certain variables at the beginning and end of restricted thrust regions, it is convenient to write equation (6-9) as

$$-\underline{\xi} = \sum_{i=0}^{j-1} \frac{\partial}{\partial \underline{\nu}} \int_{t_i}^{t_{i+1}} \Lambda^* \, B \, B^T \Lambda^{*T} \gamma \underline{\nu} \, dt \, \Delta \underline{\nu} \qquad (6\text{-}10)$$

where j is the number of distinct phases of the trajectory. Expanding (6-10) according to Leibniz's rule yields

$$-\underline{\xi} = \sum_{i=0}^{j-1} \left[\int_{t_i}^{t_{i+1}} \left(\Lambda^* \, B \, B^T \Lambda^{*T} \gamma + \Lambda^* \, B \, B^T \Lambda^{*T} \underline{\nu} \frac{\partial \gamma}{\partial \underline{\nu}} \right) dt \right.$$

$$\left. + \Lambda^* \, B \, B^T \Lambda^{*T} \gamma \underline{\nu} \, \Big|_{t_i}^{t_{i+1}} \frac{\partial t_k}{\partial \underline{\nu}} \right] \Delta \underline{\nu} \qquad (6\text{-}11)$$

$$(k = i, \; i+1)$$

Since γ is the discontinuous switching parameter, the time at which a discontinuity occurs, t_k and its derivative with respect to $\underline{\nu}$ may be evaluated from the derivation in Chapter IV. Rewriting equation (3-76) we obtain

$$\frac{\partial t_k}{\partial \underline{\nu}} = - \frac{\dfrac{\partial q}{\partial \underline{\nu}}}{\dfrac{\partial q}{\partial t}} \qquad (6\text{-}12)$$

By comparing (3-52) and (6-2) observe that

$$q = |\underline{\lambda}_v| \qquad (6\text{-}13)$$

Using (6-2) and (6-3) and taking the derivative, yields

$$\frac{\partial t_k}{\partial \underline{\nu}} = - \frac{\underline{\lambda}_v^T \, B^T \Lambda^{*T}}{\underline{\lambda}_v^T \, \underline{\lambda}_r} \tag{6-14}$$

The term $\partial\gamma/\partial\underline{\nu}$ may be evaluated from the definitions in Chapter III, which are written here in more convenient form.

$$\gamma = \min \left[1; \frac{a_o}{m} \frac{1}{|\underline{\lambda}_v|} \right] \quad \text{(VSI)} \tag{6-15}$$

$$\gamma = \left\{ \begin{array}{ll} 0 \quad \text{if} \quad \dfrac{2a_o}{m} \dfrac{1}{|\underline{\lambda}_v|} \geq 1 \\ \dfrac{a_o}{m} \dfrac{1}{|\underline{\lambda}_v|} \quad \text{if} \quad \dfrac{2a_o}{m} \dfrac{1}{|\underline{\lambda}_v|} < 1 \end{array} \right\} \text{(CSI)} \tag{6-16}$$

Therefore by differentiating and applying equation (6-2)

$$\frac{\partial\gamma}{\partial\underline{\nu}} = \left\{ \begin{array}{ll} \dfrac{a_o}{m} \dfrac{\underline{\lambda}_v^T \, B^T \Lambda^{*T}}{|\underline{\lambda}_v|^3} & \text{if } \gamma \neq 0, 1 \\ 0 & \text{if } \gamma = 0, 1 \end{array} \right\} \tag{6-17}$$

From equation (6-6) let us make the following definitions for $\dot{\underline{\eta}}$ and $\dot{M}$.

$$\dot{\underline{\eta}} = \gamma \Lambda^* \, B \, B^T \Lambda^{*T} \underline{\nu} \tag{6-19}$$

$$\dot{\underline{\eta}} = \dot{M} \, \underline{\nu} \tag{6-20}$$

Using (6-11) and (6-14) through (6-20) one obtains

$$- \underline{\xi} = \int_0^{t_f} \left(\dot{M} - \frac{\gamma' \, \dot{\underline{\eta}} \, \dot{\underline{\eta}}^T}{\left(\frac{a_o}{m}\right)^2} \right) dt \, \Delta\underline{\nu} - \left. \frac{\dot{\underline{\eta}} \, \dot{\underline{\eta}}^T}{\gamma |\underline{\lambda}_v^T \, \underline{\lambda}_r|} \, \Delta\underline{\nu} \right|_{t_{\text{cut on}} + \epsilon}^{t_{\text{cut off}} - \epsilon} \tag{6-21}$$

where we let

$$\gamma' = \left\{ \begin{array}{ll} 0 & \gamma = 1, 0 \\ \gamma & \gamma \neq 1, 0 \end{array} \right\} \tag{6-22}$$

The final term in equation (6-21) is evaluated at points such that γ is not zero, i.e. ϵ before cut-off and ϵ after cut-on.

Define

$$M^* = \int_0^{t_f} \left(\dot{M} - \frac{\gamma' \dot{\underline{\eta}} \dot{\underline{\eta}}^T}{\left(\frac{a_o}{m}\right)^2} \right) dt - \frac{\dot{\underline{\eta}} \dot{\underline{\eta}}^T}{\gamma \left| \underline{\lambda}_v^T \underline{\lambda}_r \right|} \Bigg|_{t_{on}}^{t_{off}} \tag{6-23}$$

then

$$\Delta \underline{\nu} = - M^{*-1} \underline{\xi} \tag{6-24}$$

and from (6-4), (6-5) and (6-6)

$$\nu^c = M^{-1} \eta - M^{*-1} \underline{\xi} \tag{6-25}$$

Thus the correction procedure to be employed is

$$\underline{\nu}_n = \underline{\nu}_{n-1} - M^{*-1} \underline{\xi} \tag{6-26}$$

6.4 Quantities to be Computed

In order to obtain the quantities M^* and $\underline{\xi}$ of equation (6-26) it is necessary to integrate the state variables, the adjoint set, the vector $\dot{\eta}$ and the matrix M^*. In addition it is convenient to integrate the Euler variables rather than compute them from other quantities.

From the discussion in Chapter V it may be noted that the adjoint equations are to be integrated backward in time from $\Lambda_f = I$. However, it is easily shown that integrating backwards in time is an unnecessary complication. In Chapter III the variational equations of state and their adjoint set were solved together to produce

$$\Lambda_f \, \delta \underline{s}_f = \Lambda_t \, \delta \underline{s}_t + \int_t^{t_f} \Lambda B \, \delta \underline{a} \, dt \tag{6-27}$$

and Λ_f was arbitrarily chosen equal to the identity in the guidance problem. For computing the entire trajectory from $t = 0$ to $t = t_f$ one may write an equivalent equation

$$-\Lambda_f^{*} \, {}^* \underline{\xi} = \Lambda_o^* \, \delta \underline{s}_o + \int_0^{t_f} \Lambda^* \, B \, B^T \, \Lambda^{*T} \Delta (\gamma \, \nu) \, dt \tag{6-28}$$

where Λ_f^{**} has both the last row and last column of Λ deleted in order to be compatable with $\underline{\xi}$ and where $\delta \underline{s}_o = \underline{O}$ since the problem starts from a given state. Since the initial condition on Λ is still arbitrary it is convenient to choose $\Lambda_o = I$ so that backward integration is not necessary. Thus in place of (6-24) use

$$-\Lambda_f^{**} \underline{\xi} = M^* \Delta \underline{\nu} \tag{6-29}$$

$$\Lambda_o = I \tag{6-30}$$

An interpretation of this procedure is as follows. An arbitrary acceleration program is selected (i. e. estimate $\underline{\nu}$) ; then the state and adjoint equations are integrated forward resulting in a miss in position and velocity, - $\underline{\xi}$. Since the acceleration program satisfies the conditions for an optimum to whatever terminal point it reached, namely $\left(\begin{Bmatrix} \underline{r}_T \\ \underline{v}_T \end{Bmatrix} + \underline{\xi} \right)$, the propellant required to reach that point is minimum. Thus we are assured that when $\underline{\xi} \to \underline{O}$, $m_f \to m_{f(max)}$ also. It is unimportant, therefore, that the "miss" in m_f is unknown, since if the procedure converges on the physical boundary conditions it does so via a propellant-optimal trajectory.

The term $-\Lambda_f^{**} \underline{\xi}$ merely reflects the physical miss back to the initial state and compensates for the procedure of integrating the adjoint set forward in time instead of backward in time. In the machine procedure it is convenient to use equation (6-31) in lieu of (6-26).

$$\underline{\nu}_n = \underline{\nu}_{n-1} - M^{*-1} \Lambda_f^{**} \underline{\xi} \tag{6-31}$$

The full set of equations that must be integrated are listed with the appropriate initial conditions.

$$\dot{\underline{s}} = \underline{g} (\underline{s}, \underline{a}) \tag{6-32}$$

$$\underline{s} (0) = \underline{s}_o \tag{6-33}$$

$$\dot{\Lambda} = -\Lambda A \tag{6-34}$$

$$\Lambda_o = I \tag{6-35}$$

$$\dot{\underline{\lambda}} = - A^T \underline{\lambda} \tag{6-36}$$

$$\underline{\lambda}(0) = \underline{\nu} \tag{6-37}$$

$$\dot{M}^* = \gamma \Lambda^* B B^T \Lambda^{*T} - \frac{\gamma' \dot{\underline{\eta}} \dot{\underline{\eta}}^T}{\left(\frac{a_o}{m}\right)^2} \tag{6-38}$$

$$M^*(0) = O \tag{6-39}$$

$$\dot{\underline{\eta}} = \dot{M} \underline{\nu} \tag{6-40}$$

where M is the first term of M^*

$$M(0) = O \tag{6-41}$$

$$\underline{\eta}(0) = \underline{O} \tag{6-42}$$

An information flow chart and the FORTRAN program used to mechanize the preceding equations are included in Appendix G.

The procedure worked satisfactorily, converging to very small values for $\underline{\xi}$ with surprisingly few iterations. The initial estimates for $\underline{\nu}$ that were used are

$$\underline{\nu} = \underline{O} \qquad \text{(VSI trajectories)} \tag{6-43}$$

$$\underline{\nu} = \begin{Bmatrix} \underline{O}_3 \\ 0 \\ a_o \\ 0 \end{Bmatrix} \qquad \text{(CSI trajectories)} \tag{6-44}$$

CHAPTER VII

A NUMERICAL EXAMPLE

7.1 Summary of Chapter VII

In this chapter a sample heliocentric transfer from Earth to Mars is described. The selected mission was programmed for machine computation to test the theories developed in the thesis. Numerical results from the test are presented and discussed.

7.2 General Remarks

The linear guidance theory of Chapter III assumes that an optimal reference trajectory is known. In order to test the guidance theory, therefore, it is necessary to compute an optimal reference trajectory. In Chapter V the discussion of a superfluous Euler equation shows that the solution of the linear guidance equation satisfies the necessary conditions for an optimum. Thus in theory, iterative solution of the guidance equation will generate the desired optimal reference. If the technique converges the linear guidance theory is validated. If the procedure converges rapidly then the theory also provides a simple, fast and effective way of generating low-thrust trajectories.

The procedure was found to converge rapidly for both CSI and VSI vehicles.

7.3 The Mission

The mission selected was a 150-day heliocentric transfer from Earth to Mars. Flight time of 150 days was chosen in order that numerical results could be compared with a 162-day coplanar trajectory generated by Friedlander[12]. The period chosen corresponds to the favorable opposition of Mars during the summer of 1971[37]. The departure date, from a position on Earth's sphere of influence, is J. D. 244-1090.5. Arrival at Mars sphere of influence is 150 days later. The vehicle is assumed to have Earth's orbital velocity at departure and to match the Martian velocity at arrival. These velocity conditions are

not optimal for the entire mission, i.e. escape, transfer, capture, however the author wanted to test out-of-plane components of the state vector. The use of planetary velocities for initial and final velocity permitted this.

Both CSI and VSI modes of control were tested. The assumption was made that a space vehicle of given size and power could be controlled under either mode of operation, CSI or VSI. Thus two complete sets of data were generated which differ only in mode of control. In addition the adjoint sets for an optimal thrust program and an optimal acceleration program were generated for each control mode. The purpose of having both types of adjoint functions is to permit all engine anomalies of interest to be studied. The subtle difference between the two sets of adjoint functions is discussed in Appendix C.

7.4 Computational Coordinates

The coordinate system used for the mission analysis is a simple but effective system which is defined by the transfer plane. The transfer plane is the plane which passes through the sun and contains the desired departure and arrival points. The x axis passes through the launch point, the z axis is normal to the plane in the northerly direction, and the y axis completes the right hand triad. A procedure for transforming ephemeris data into computational coordinates is derived in Appendix F. Figure 7-a illustrates the geometry of the transfer.

7.5 Engine Selection

Engine sizing for the sample mission was computed on the basis of the mass distribution for maximum payload derived in Chapter II. That is

$$\frac{\beta_{opt}}{\alpha} = \frac{p_{opt}}{m_o} = \sqrt{\frac{J}{\alpha}} - J \tag{7-1}$$

A trial trajectory for a mass independent VSI rocket was computed to obtain a first approximation for J. The value for the one-way transfer was scaled up by a factor of 4 to provide a more realistic value for a round trip mission; also to insure that mass would never be reduced to zero during computer tests. A value for α of 10 kg/kw was selected as

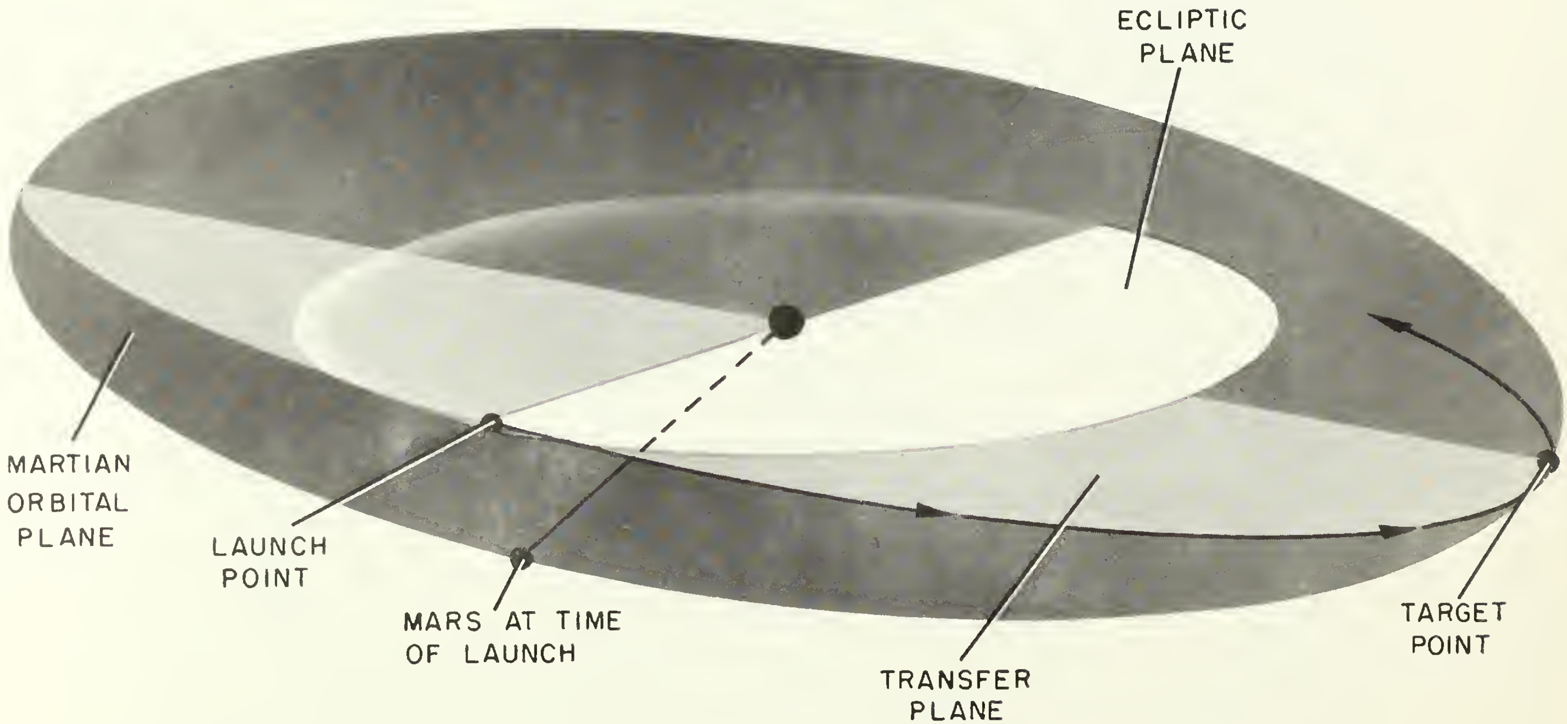

Fig. 7-a. Geometry of the sample transfer.

being reasonable, though perhaps unattainable in less than a decade. The resulting value for exhaust power is

$$\frac{P}{m_o} = 0.0242 \frac{kw}{kg} \qquad (7\text{-}2)$$

This value for power was used for both modes of control, (VSI) and (CSI).

The VSI trajectory was generated and the resulting value for initial acceleration, $1.57 \times 10^{-4}\ g_o$, was used in the field-free space equations of Appendix A to obtain an optimum initial acceleration (thrust level) for the CSI vehicle. A value of $1.2 \times 10^{-4}\ g_o$ was used. This value of optimum initial acceleration, based on field-free space, is in surprisingly good agreement with the more rigorous computations of Melbourne and Sauer.

7.6 Numerical Results

The computer output data for the final iteration of each mode of control are reproduced as Appendix H. In addition, plots of several interesting output quantities are presented as Figures H-a through H-r. The data samples and the plots are strictly valid only for the particular cases they represent, however they are indicative of the order of magnitudes applicable to many one-way missions in the solar system. The adjoint functions, for example, (Figures H-d through H-r) are in good agreement with the values obtained by Friedlander for a coplanar transfer.

It is interesting to observe that the acceleration magnitude for a VSI vehicle, Figure H-a, is approximately a linear function of time and almost symmetric with respect to the midpoint. Comparison with the field-free space acceleration program, Figure 2-c, which is linear and symmetric, confirms that to first order analysis of VSI trajectories in field-free space may be applied to the gravitational field.

This result only confirms the work of other investigators and was anticipated. The analysis for CSI vehicles in field-free space is not as straightforward and requires a large amount of work. The field-free space derivations in Appendix A result in a value for the optimum

CSI thrust level which is in good agreement with the computer results of Melbourne and Sauer. The value obtained for the final mass ratio is larger than the value which results from computer studies in the gravitational field. Referring to Figure 2-d, observe that the point representing the CSI transfer falls slightly below the curve representing field-free space prediction. This is satisfying in that the penalty for CSI control is less than expected. However, before concluding that field-free space is always a good predictor for CSI transfers, it will be necessary to test a large number of additional points to determine if agreement is sufficiently good to warrant the computational effort. The limitation of time has prevented the author from undertaking such a study.

In Figures H-b and H-c the transfer plane components of position and acceleration are plotted. The out-of-plane components are quite small and are not considered further. A surprising result is that the maximum difference between the physical paths of the VSI and CSI trajectories is less than 0.01 A u. The large difference in the form of the acceleration programs, Figure H-a, leads one to anticipate rather large trajectory differences. This is found not to be the case.

A very important characteristic of continuous-thrust rockets is shown in Figure H-p. This is the sensitivity of position error to mass change when the thrust program is specified. The y component of position at arrival time is subject to an error of 0.81×10^{6} km for 1% variation in launch weight. For a one-ton vehicle this is approximately equivalent to a 25,000 mile terminal error for each pound of launch weight variation. The large sensitivity to mass changes emphasizes the requirement for accelerometers with sensitivities of the order of 10^{-5} g. Such instruments will be extremely important in the navigation of low-thrust spacecraft.

7.7 Convergence of the Computation Routine

The results discussed in the preceding section, although interesting, are only by-products of the experiment devised to test linear guidance theory as a computational method for trajectories. Convergence of the routine is crucial to the thesis since the linear theory

purports to null the miss vector for small variations in state. Application of the guidance theory to the launch state, i. e. trajectory computation, provides the most severe test of guidance. The results of this test are plotted in Figure 7-b. The criterion chosen as indicative of convergence is the RMS value of miss vector length. In order to be dimensionally compatable, the velocity components are multiplied by flight time, 150 days. Therefore the ordinate in Figure 7-b is the RMS value in A. u., of $\begin{Bmatrix} \underline{\xi}_r \\ t_f \underline{\xi}_v \end{Bmatrix}$. The abscissa is the iteration number. Several cases were tested; the three plotted are indicative of all tests.

The first one or two points in each case represent the result of estimating initial values for the Euler variables, $\underline{\nu}$, and were widely scattered. These points are not plotted.

The unrestricted VSI trajectory converges very rapidly; reducing $|\underline{\xi}|$ by approximately two orders of magnitude each iteration. Convergence is very smooth. The CSI cases converge less rapidly and less smoothly, but satisfactorily. In general, as the thrust limit, a_o, is reduced, convergence is slower. If, for example, a_o is reduced ϵ below the value which permits no coast period, the procedure will not converge. This physically corresponds to a vehicle "in extremis" such that insufficient thrust is available to complete the mission. The result also indicates that reserve power must be available for guidance. A CSI vehicle which is beyond the coast phase cannot correct for state error in all six components of $\underline{\xi}$ unless reserve power is available. If reserve power is not available, guidance must be based on a formulation such as Pfeiffer's[14], which gives the minimum miss.

It should be reported that the CSI miss vector magnitude entered a small limit cycle around the target prior to addition of the term $\partial t_k / \partial \underline{\nu}$. This term is derived in section 6. 3. Prediction of the switching time by including the above derivative is necessary to obtain the desired accuracy.

It may be desirable to include an additional term in the procedure to smooth convergence of the CSI routine. The term $\partial m / \partial \underline{\nu}$, evaluated

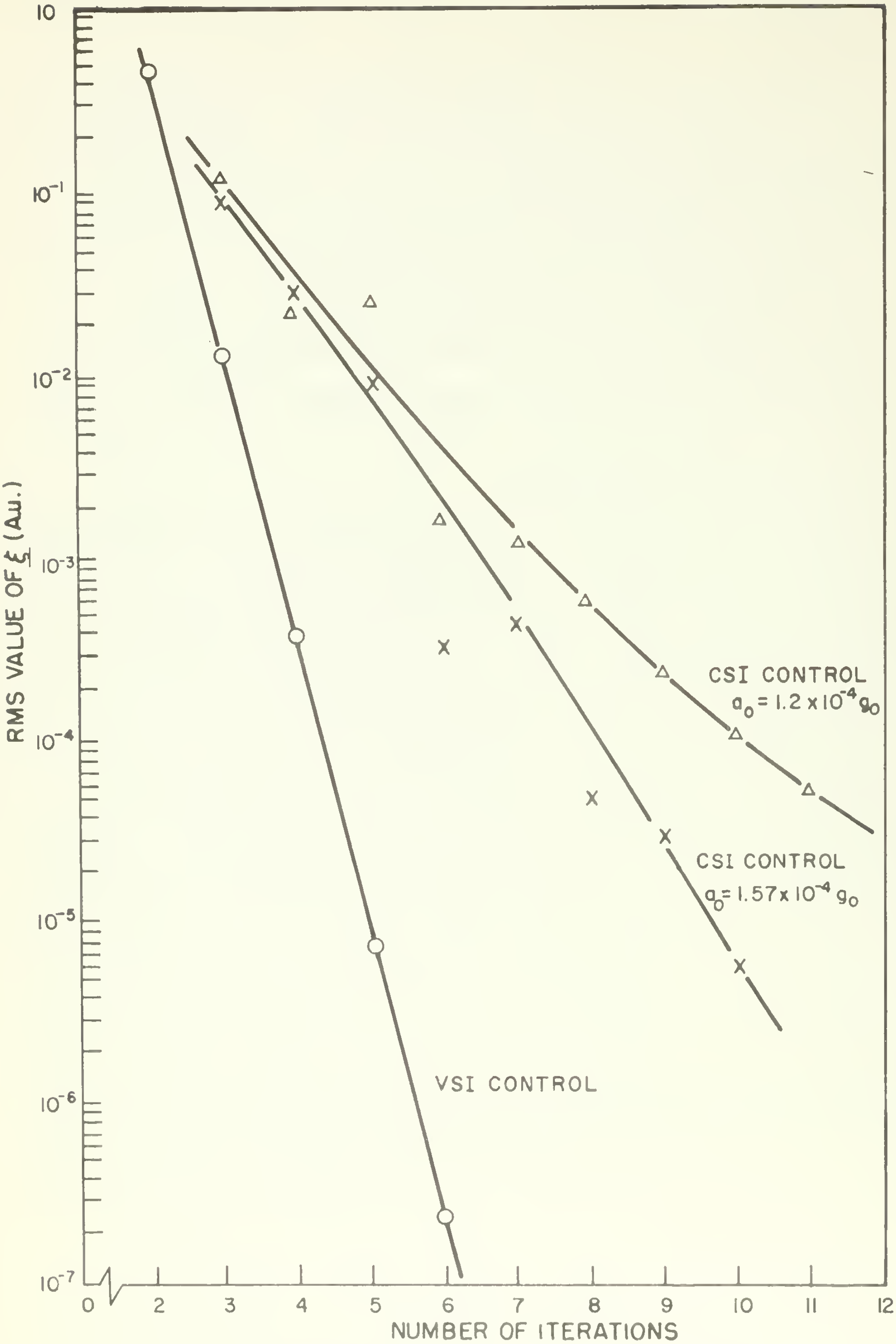

Fig. 7-b. Convergence of computation routine.

at the switch points, will probably be sufficient to do this. Computer tests were completed before the significance of this term was recognized and time limitation precluded retesting of the procedure.

The accuracy obtained from the tests is significant. The routine was stopped after a predetermined number of iterations to conserve computer time, however for VSI trajectories, the largest error in a component of position was of the order of one mile. The largest error in velocity was of the order of 20 feet per hour. These values are given in the computer results in Appendix H. The computer was stopped before attaining this accuracy for CSI trajectories, however the convergence plots in Figure 7-b indicate such accuracy is attainable with a sufficient number of iterations.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

8.1 Summary

In this report the objective has been to justify the argument: "There exists a linear method which produces a propellant-optimal control program in a noniterative form for guidance of low-thrust space vehicles and which provides a simple, rapidly converging iterative technique for computing propellant-optimal trajectories."

In pursuit of this objective it is necessary, in Chapter II, to develop the parameters which characterize low-thrust vehicles. Due to "state of the art" restrictions on variable-specific-impulse (VSI) machines, the parameters for constant-specific-impulse (CSI) vehicles are developed in addition to the idealized mode of control.

As an adjunct of Chapter II, the equations for field-free space application of CSI control are derived in Appendix A. This derivation is used to test the validity of approximating CSI transfers in the solar system by field-free space analysis, as has previously been done for constant-power vehicles.

The parametric derivations provide a starting point for investigating propellant-optimal guidance of low-thrust vehicles. For this study the vehicle is characterized by its "state". The state is represented by a vector consisting of three components of position, three components of velocity and the vehicle mass. The differential equation of state is linearized by considering variations of the state vector with respect to an optimal reference trajectory. Auxiliary functions, called adjoint variables, are used to solve the linearized differential equation. The solution to the expanded set of equations is called the state transition equation or fundamental guidance equation and serves as the basis of the guidance theory.

The linear guidance theory is developed in Chapter III and applied to both modes of vehicle control. The solution for an unrestricted VSI vehicle is derived using the calculus of variations and then using Pontryagin's maximum principle to prove that both methods lead to the same result. The CSI problem is then solved using the Hamiltonian. The crucial step in either approach is deletion of the adjoint functions associated with the mass rate equation. Deletion of this set of functions to form a reduced adjoint set permits a solution to be obtained directly. Otherwise, the system is indeterminate due to a singular matrix. Proof of the singularity and justification for deleting the one set of adjoint functions are treated in Appendix C and Chapter V respectively.

As an adjunct of Chapter III, explanation of the adjoint relationship and derivation of Pontryagin's principle are presented in Appendices B and D respectively. Also in Appendix D is a discussion of optimality criteria as derived from the Pontryagin principle and from the calculus of variations. Important properties of the state transition equation are presented and discussed in Appendix C.

The problem of estimating the vehicle state is studied in Chapter IV as a problem in navigation. The method of redundant measurements studied by Battin, Stern and Potter is extended to include continuous measurement of low-thrust engine performance. Based on the concept of filtering redundant data with a biased filter, two methods are derived for predicting future state. One is a simple method for short term prediction; the other is more complex but also more accurate for long range prediction. The effect of omitting engine measurements is discussed.

The problem of computing optimal reference trajectories is discussed in Chapters V and VI. By first deriving the Euler equations with the calculus of variations and then demonstrating that the set contains a superfluous Euler equation, the linear guidance theory is shown to be useful for computing optimal reference trajectories. The superfluous Euler variable is a result of optimizing a state variable.

To provide a comparison of linear guidance with better known computation techniques, the steepest ascent formulation of the trajectory problem is presented in Appendix E.

Chapter VII describes the sample mission that was simulated on a digital computer to test the guidance equation as a computational device for trajectories. The results of the test are discussed.

8.2 Conclusions

On the basis of the analysis and the subsequent simulation the following conclusions were reached.

1. The linear guidance method is applicable to guidance of low-thrust vehicles in interplanetary flight.
2. The rapid convergence of the test procedure proves the method to be applicable for computing propellant-optimal trajectories.
3. For CSI transfers, reserve power for guidance is necessary.
4. Mathematical descriptions of many optimization problems which contain a state variable in the cost, will produce a superfluous Euler variable when the classical calculus of variations is used. Deletion of this Euler variable removes several difficulties associated with solving optimal control problems.
5. Sightings on celestial bodies at discrete intervals may be combined with continuous measurements of engine performance to estimate the state of the vehicle.
6. The uncertainty in state is increased if engine measurements are omitted.
7. Field-free space analysis of CSI transfers provide a reasonable approximation for the results in a gravitational field, however additional study is needed to ascertain the general applicability of the method.

8.3 Contributions of the Investigation

The items in the report which are believed to be novel are discussed in this section.

Derivation of the propellant-optimal control law by the method of Chapter III is thought to be original. The crucial step in the derivation is the formation of a reduced adjoint set. This permits inversion of a matrix which otherwise would be singular. Other approaches to this problem usually require the addition of artificial constraints or weighting matrices which remove the singularity but also change the character of the cost function. The method of this thesis preserves the cost function and produces the optimal control directly.

The explanation of the singularity on the basis of a superfluous Euler variable is thought to be novel. The simplification resulting from deletion of the extra variable, by using the reduced adjoint set, may resolve many difficulties associated with optimal control problems other than the one in this report.

The equations which combine the celestial sights and the continuous measurements of engine performance are not believed to have been previously derived.

Finally, the method of generating the optimal trajectory is believed to be simpler than methods previously used. The method results from linearizing the state equations but using the complete nonlinear cost function.

8.4 Recommendations for Further Study

The current research has revealed some areas where additional work might produce fruitful results and other areas where information is lacking with respect to low-thrust transfers.

The computational method needs to be expanded to the many body problem. Preliminary work does not show any monumental difficulties associated with such an effort. The primary problem is to include an ephemeris in the routine.

It appears possible to use the method as a search routine for finding families of trajectories, with the goal of defining optimal launch times for low-thrust missions.

A study of methods to join different segments of an optimal trajectory should be interesting. In particular the idea of making only one coordinate change between launch and capture is appealing.

Research to define rigorously the concept of superfluous Euler variables is needed. It is doubtful that the singularities in all optimization problems can be removed by the method of this thesis, but perhaps some can. Knowledge of the general applicability is needed.

When data on the reliability of low-thrust power plants and thrusters is available, a study of the effects of engine anomalies on the probability of mission success should be made.

A study of VTA guidance using the "critical" plane associated with ballistic guidance is needed[31]. This concept is difficult to visualize in the six dimensional phase space associated with the equations of motion. However, the existence of a concept analogous to the "critical plane" idea might provide additional insight to the guidance problem.

APPENDIX A

TRANSFERS IN FIELD-FREE SPACE

A. 1 Introduction

In this appendix the equations for point-to-point transfer of low-thrust rockets in field-free space (FFS) are presented. The optimal transfer problem for a constant power VSI rocket is a straight forward application of the calculus of variations. Examples of this transfer are found frequently in the literature. It is reproduced here for completeness.

Point-to-point transfer of a CSI rocket is more difficult to compute. The equations are derived in this appendix. Results of the derivation were applied to the sample mission in the thesis to ascertain the validity of using FFS predictions in the gravitation field. FFS analysis is found to furnish a reasonable approximation for CSI transfers, however the analysis is quite tedious.

The author is indebted to Mr. Neal Carlson for checking the derivation and for suggesting different methods of approach.

A. 2 Constant Power Transfer

Assume that in FFS we desire to traverse the distance L in the time T such that the vehicle begins and ends at rest and such that the acceleration integral is a minimum.

That is

$$\text{minimize } J = \int_0^T \frac{a^2}{2}\, dt \tag{A-1}$$

$$\text{subject to } \int_0^T v\, dt = L \tag{A-2}$$

and the boundary conditions

$$\begin{aligned} v\,(0) &= 0 \\ v\,(T) &= 0 \end{aligned} \tag{A-3}$$

Forming a functional F, one obtains

$$F = \int_0^T \frac{\dot{v}^2}{2}\, dt + \pi \left(L - \int_0^T v\, dt\right) \qquad \text{(A-4)}$$

Direct application of the variational technique produces the optimal velocity schedule from which the acceleration is easily derived.

$$a = \frac{6L}{T^2} \left(1 - \frac{2t}{T}\right) \qquad \text{(A-5)}$$

Evaluating the acceleration integral, obtain

$$J = \frac{6L^2}{T^3} \qquad \text{(A-6)}$$

From equation (A-5) the optimal initial acceleration is

$$a_o = \frac{6L}{T^2} \qquad \text{(A-7)}$$

A plot of equation (A-5) is presented in the discussion of Chapter II as Figure 2-c.

A. 3 Constant-Specific-Impulse Transfer

The well known equations for a conventional rocket must be integrated in the analysis of a CSI transfer. That is

$$v = c \ln MR \qquad \text{(A-8)}$$

Since the variational approach is degenerate for this type of optimization problem, we shall simply apply the known result from the Hamiltonian approach of Chapter III, that thrust is either full on or full off for CSI vehicles in a linear field. Integration of equation (A-8) is over the initial and final thrusting periods with a coast in between. That is

$$L = \int_0^{t_1} v(t)\, dt + \int_{t_2}^{T} v(t)\, dt + v(t_1)(t_2 - t_1) \qquad \text{(A-9)}$$

subject to the boundary conditions (A-3). This rather lengthy computation reduces to

$$L = c\,(t_2 - t_1)\ \ln\left(\frac{1}{1 - \dot{m}t_1}\right) + a_o t_1^{\,2} \tag{A-10}$$

where a_o in the initial acceleration and $\dot{m}$ is the normalized flow rate (i.e. $m_o = 1$). Some additional algebraic manipulation allows (A-10) to be cast in the form

$$\frac{\dot{m}L}{c} = -\left(\dot{m}T + m_1^2 - 1\right)\ln m_1 + (1 - m_1)^2 \tag{A-11}$$

where

$$m_1 = 1 - \dot{m}t_1 \tag{A-12}$$

From the work of Chapter II observe that for $\dot{m}$ a positive number

$$a_o = \dot{m}c \tag{A-13}$$

$$p = +\frac{1}{2}\,\dot{m}c^2 \tag{A-14}$$

Thus

$$\dot{m} = \frac{a_o^{\,2}}{2p} \tag{A-15}$$

$$c = \frac{2p}{a_o} \tag{A-16}$$

Equation (A-11) may now be written in terms of initial acceleration, power, mass, and the dimensions of the problem, L and T.

$$\frac{a_o^3\,L}{4p^2} = -\left(\frac{a_o^2\,T^2}{2p} + m_1^2 - 1\right)\ \ln m_1 + (1 - m_1)^2 \tag{A-17}$$

In the VSI case a_o was $\frac{6L}{T^2}$. Let the CSI initial acceleration be an unknown multiple of that value

$$a_o = x\,\frac{6L}{T^2} \tag{A-18}$$

Further, define a parameter R.

$$R \equiv \frac{18L^2}{pT^3} \tag{A-19}$$

Substituting (A-18) and (A-19) into (A-17) one obtains

$$\frac{1}{6} R^2 x^3 = -\left(R x^2 + m_1^2 - 1\right) \ln m_1 + (1 - m_1)^2 \tag{A-20}$$

From the condition that the velocity change must be the same for both thrusting intervals if boundary conditions are satisfied, the final mass ratio is

$$MR = \left(\frac{1}{m_1}\right)^2 \tag{A-21}$$

Therefore if m_1 is a maximum, mass ratio is a minimum and likewise for the acceleration integral.

Using equation (A-20), solve for x such that $\frac{\partial m_1}{\partial x} = 0$. The result is

$$\frac{\partial m_1}{\partial x} = 0 \tag{A-22}$$

when

$$R\, x \left(\frac{1}{2} R\, x + 2 \ln m_1\right) = 0 \tag{A-23}$$

$x = 0$ is a trivial solution and the desired solution is

$$x = -\frac{4}{R} \ln m_1 \tag{A-24}$$

$$x = \frac{2}{R} \ln MR \tag{A-25}$$

If equation (A-24) is substituted into (A-20) the result is

$$-\frac{16}{3R} (\ln m_1)^3 + (1 - m_1^2) \ln m_1 + (1 - m_1^2) = 0 \tag{A-26}$$

The transcendental equation (A-26) is the relation between m_1 and R for optimal CSI operations in field-free space. The optimum initial

acceleration is given by (A-24) or (A-25). Equation (A-26) is plotted in Figure 2-d. It is observed that CSI control always uses more propellant than VSI control. Also plotted in Figure 2-d is the CSI curve for $a_o = \frac{6L}{T^2}$ (i.e. $x = 1$). This value of a_o is considerably more expensive than a transfer with $a_o = a_{o\,(opt)}$. Figure A-a contains plots of the required coast time for CSI control. These are obtained by solving equation (A-10) for $(t_2 - t_1)/T$ in terms of m_1 and R. Again the work involved is substantial.

Observe that for the case where

$$m_1 = 1 - \epsilon \tag{A-27}$$

where ϵ is a small quantity, equation (A-27) may be substituted into (A-26) and the resulting expression solved for m_1 by neglecting higher order terms.

$$m_1 = 1 - \frac{3R}{16} \tag{A-28}$$

By approximating $\ln m_1$ as

$$\ln m_1 = -\epsilon \; (1 + 1/2\,\epsilon + \cdots\cdots) \tag{A-29}$$

one obtains

$$x_{opt} = 3/4 \tag{A-30}$$

Therefore, for transfers such that the propellant consumption is small with respect to the total initial mass, the optimum initial acceleration for CSI vehicles is approximately three-fourths of the corresponding optimum initial acceleration for VSI vehicles.

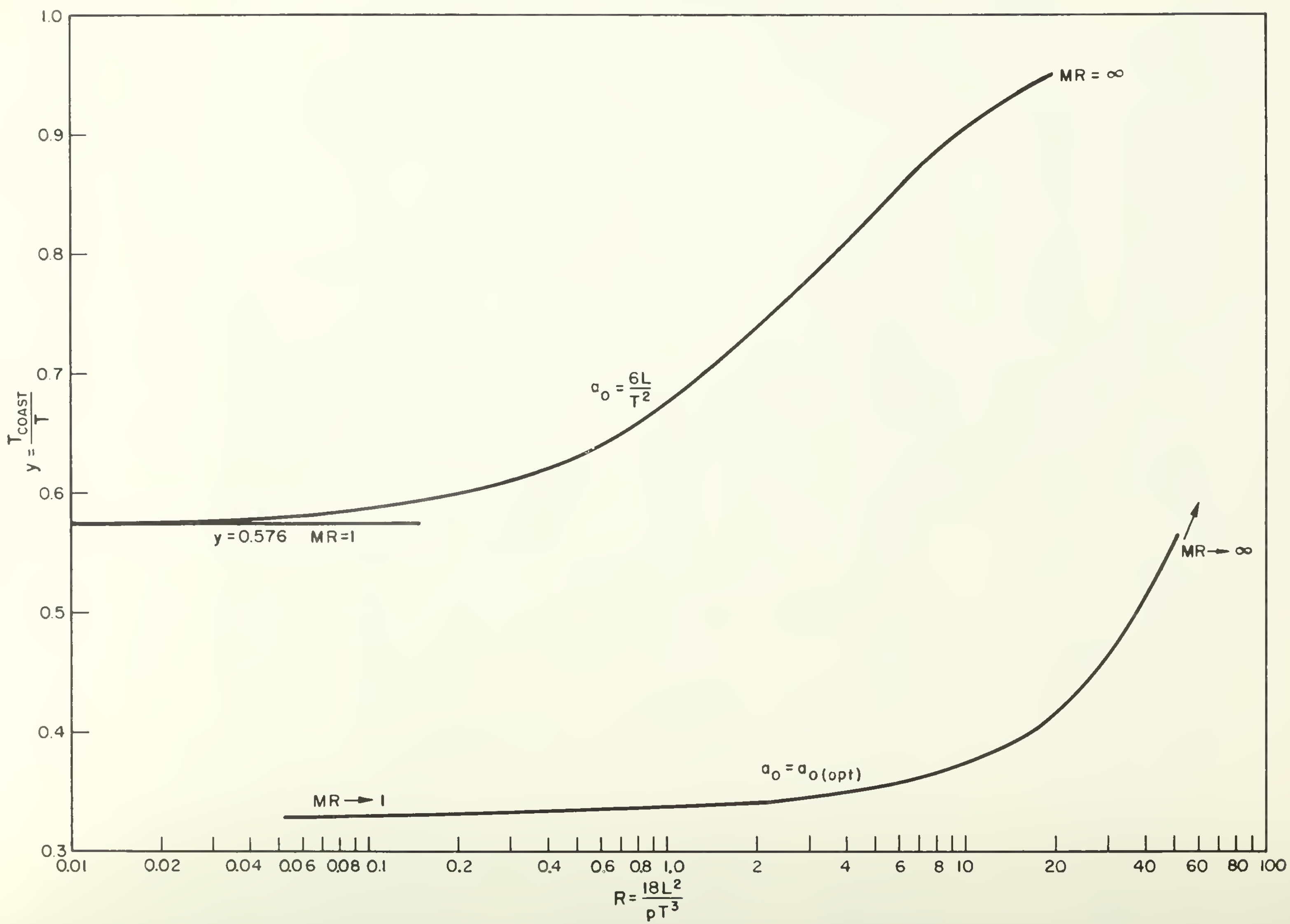

Fig. A-a. Coast time for constant-specific-impulse rocket in field-free space.

APPENDIX B

ADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS

B. 1 Introduction

The purpose of this appendix is to review briefly the concept of the adjoint system of differential equations and to illustrate its use in a simple guidance problem. Although the concept is not new, it has not been used as a standard engineering technique in guidance and control problems until very recently. For solution of the two-point boundary value problem which occurs in the navigation and guidance of spacecraft however, the method of adjoints is a particularly useful tool.

The development in this section is not intended to be rigorous and exhaustive but merely illustrative of the method used throughout the thesis. The material is taken primarily from the lecture notes of Professor Frank D. Faulkner of the United States Naval Postgraduate School.

B. 2 Method of Adjoints

Consider the following ordinary scalar differential equation,

$$\ddot{x} + 3\dot{x} + 2x = f(t) \tag{B-1}$$

which may be used to define the operator

$$L(x) = \left(\frac{d^2}{dt^2} + \frac{3d}{dt} + 2\right) x = f(t) \tag{B-2}$$

where f(t) is an arbitrary but known function. In a manner closely related to the method of variation of parameters, form the integral

$$J = \int_0^T \Lambda L(x)\, dt = \int_0^T \Lambda f(t)\, dt \tag{B-3}$$

where Λ is an unspecified function which will be chosen later to satisfy certain boundary conditions in addition to a functional relationship. Integrating equation (B-3) by parts to eliminate the derivative of x from the integrand, one obtains

$$J = (\Lambda x + 3 \Lambda x - \dot{\Lambda} x) \Big|_0^T + \int_0^T x (\ddot{\Lambda} - 3 \dot{\Lambda} + 2\Lambda) \, dt \tag{B-4}$$

If Λ is chosen such that it satisfies

$$L^* (\Lambda) = (\frac{d^2}{dt^2} - \frac{3d}{dt} + 2) \Lambda = 0 \tag{B-5}$$

then the definite integral J is a function only of the first term on the right side of equation (B-4). The operation of integrating by parts to eliminate the dependent variable from the integrand defines the adjoint operator L^*. If $L^* = L$ the system is said to be self adjoint and has some very useful properties. However, these are of no concern at the moment.

A general solution to the adjoint equation (B-5) is

$$\Lambda = C_1 e^t + C_2 e^{2t} . \tag{B-6}$$

Suppose that the desired quantities are $\dot{x}(T)$ and $x(T)$ and that $\dot{x}(0)$ and $x(0)$ (i. e. two constants of integration associated with the original differential equation) are known. Equation (B-4) may be rewritten as

$$[\Lambda \dot{x} + (3\Lambda - \dot{\Lambda}) x]_{t=T} = [\Lambda \dot{x} + (3 \Lambda - \dot{\Lambda}) x]_{t=0} + \int_0^T \Lambda f(t) \, dt \tag{B-7}$$

If one specifies that

$$\Lambda(T) = 1 \tag{B-8}$$

$$3 \Lambda(T) - \dot{\Lambda}(T) = 0 \tag{B-9}$$

then equation (B-7) gives for $\dot{x}(T)$

$$\dot{x}(T) = [\Lambda \dot{x} + (3\Lambda - \dot{\Lambda}) x]_{t=0} + \int_0^T \Lambda f(t) \, dt \tag{B-10}$$

The constants in equation (B-6) may be evaluated from the boundary conditions of the adjoint variable, equations (B-8) and (B-9). Designate this solution Λ_1. By performing the algebra one obtains

$$\Lambda_1 = 2e^{2 (t-T)} - e^{t-T} \tag{B-11}$$

Equation (B-10) may now be solved explicitly for $\dot{x}(T)$.

$$\dot{x}(T) = (2e^{-2T} - e^{-T})\,\dot{x}_0 + 2(e^{-2T} - e^{-T})\,x_0 + \int_0^T \Lambda_1\, f(t)\, dt \tag{B-12}$$

To obtain x(T) it is necessary to specify different boundary conditions for Λ. Designate the solution satisfying these conditions as Λ_2.

$$\Lambda_2(T) = 0 \tag{B-13}$$

$$3\,\Lambda_2(T) - \dot{\Lambda}_2(T) = 1 \tag{B-14}$$

By solving equation (B-6) subject to the boundary conditions of (B-13) and (B-14) one obtains

$$\Lambda_2 = e^{t-T} - e^{2(t-T)} \tag{B-15}$$

Inserting this result into equation (B-7) yields for x(T)

$$x(T) = (e^{-T} - e^{-2T})\,\dot{x}_0 + (2e^{-T} - e^{-2T})\,x_0 + \int_0^T \Lambda_2\, f(t)\, dt \tag{B-16}$$

The preceding development illustrates the general technique to be used. However, in order to more closely approach the formulations used in the thesis, it is convenient to write the basic second order differential equation, (B-1), as two first order equations.

$$\dot{x} = y \tag{B-17}$$

$$\dot{y} = -3y - 2x + f(t) \tag{B-18}$$

Associate with the two equations two sets of adjoint functions Λ_{i1} and Λ_{i2} where the subscripts 1 and 2 refer to the associated equation and the subscript i will refer to the boundary conditions which will eventually be assigned to the Λ's. The integral J may now be formed.

$$J = \int_0^T \Lambda_{i1}(\dot{x} - y) + \Lambda_{i2}(\dot{y} + 3y + 2x)\, dt = \int_0^T \Lambda_{i2}\, f(t)\, dt \tag{B-19}$$

Integrate by parts to obtain

$$J = (\Lambda_{i1}x + \Lambda_{i2}y)\Big|_0^T - \int_0^T x(\dot{\Lambda}_{i1} - 2\Lambda_{i2}) + y(\dot{\Lambda}_{i2} - 3\Lambda_{i2} + \Lambda_{i1})dt \tag{B-20}$$

In order to eliminate the integral term it is clear that the adjoint variables must satisfy the relationship

$$\dot{\Lambda}_{i1} - 2\Lambda_{i2} = 0 \qquad \text{(B-21)}$$

$$\dot{\Lambda}_{i2} - 3\Lambda_{i2} + \Lambda_{i1} = 0 \qquad \text{(B-22)}$$

By reducing the two equations to one second order equation and solving subject to the boundary conditions

$$\Lambda_{11}(T) = 1 \qquad \text{(B-23)}$$

$$\Lambda_{12}(T) = 0 \qquad \text{(B-24)}$$

and

$$\Lambda_{21}(T) = 0 \qquad \text{(B-25)}$$

$$\Lambda_{22}(T) = 1 \qquad \text{(B-26)}$$

it is apparent that the adjoint system of equations (B-21) and (B-22) is identical with that obtained by the first formulation of the problem.

In general, if any set of first order differential equations may be written in the form:

$$\begin{Bmatrix} \dot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{Bmatrix} - A \begin{Bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(t) \end{Bmatrix} \qquad \text{(B-27)}$$

or equivalently

$$\dot{\underline{x}} - A\underline{x} = \underline{f}(t) \qquad \text{(B-28)}$$

Then there exists a set of adjoint functions satisfying the relations

$$\dot{\Lambda} + \Lambda A = O_n \qquad \text{(B-29)}$$

$$\Lambda(T) = I_n \qquad \text{(B-30)}$$

such that

$$\underline{x}(T) = \Lambda(t_1)\,\underline{x}(t_1) + \int_{t_1}^{T} \Lambda(t)\,\underline{f}(t)\,dt \qquad \text{(B-31)}$$

where $\Lambda(t)$ is the n by n matrix of adjoint functions, A is the n by n matrix of coefficients in the original set of differential equations and I_n is the n by n identity matrix.

Unfortunately, in most problems of interest the adjoint functions cannot be obtained in closed form and it is necessary to integrate the n^2 equations of (B-29) backwards in time from T to t_1 numerically, using (B-30) as the initial condition.

B. 3 Use of the Adjoint Method in a Simple Guidance Problem

In this section the set of adjoint functions for a power-limited space vehicle in two dimensional field-free space will be developed analytically. The purpose is twofold: 1) to further illustrate the technique of section B. 2, and 2) to provide a simple analytic example for use in illustrating the guidance methods expounded in the body of the thesis.

The fundamental guidance equation for thrust-limited vehicles has been defined in earlier sections as

$$\delta \underline{s}_f = \Lambda_t \, \delta \underline{s}_t + \int_t^{t_f} \Lambda B \, \delta \underline{f} \, dt \qquad \text{(B-32)}$$

For an actual interplanetary transfer the Λ matrix will be obtained simultaneously with the desired trajectory. In any event, if the trajectory is known the adjoint functions are also known or may be easily determined. To illustrate this assume a transfer in field-free space such that motion is along the positive x axis commencing from rest at the origin at t = 0. At $t = t_R$ the thrust is reversed to effect rendezvous with point x_f at time $t = t_f$. See Figure B-a.

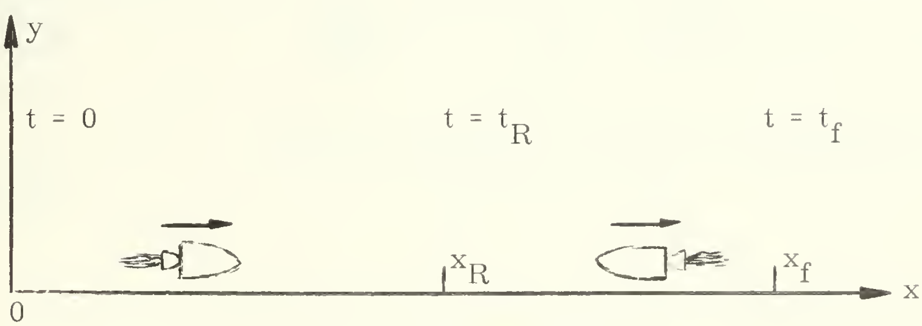

Figure B-a

The engine produces constant thrust along the nominal trajectory (which is not optimum). Presumably the engine must be controlled in some way to provide steering but it is not necessary to specify the method of control in order to determine the Λ matrix.

The equations of motion and constraint are:

$$\dot{x} = v_x \tag{B-33}$$

$$\dot{y} = v_y \tag{B-34}$$

$$\dot{v}_x = a = 2p/cm \tag{B-35}$$

$$\dot{v}_y = 0 \tag{B-36}$$

$$\dot{m} = -2p/c^2 \tag{B-37}$$

where

$$p = \frac{-\dot{m}c^2}{2} = \text{(exhaust power)} \tag{B-38}$$

$$c = \text{exhaust velocity}$$

Since the velocity and position along the nominal trajectory are easily computed as functions of time by direct integration, they may be considered as known functions and we may proceed directly to the variational equations which are of primary interest. Taking the variations of equations (B-33) through (B-38) yields

$$\delta\dot{x} = \delta v_x \tag{B-39}$$

$$\delta\dot{y} = \delta v_y \tag{B-40}$$

$$\delta\dot{v}_x = -\frac{2p}{cm^2}\delta m + \frac{1}{m}\delta\left(\frac{2p}{c}\right) \tag{B-41}$$

$$\delta\dot{v}_y = 0 \tag{B-42}$$

$$\delta\dot{m} = 0 \qquad - \delta\left(\frac{2p}{c^2}\right) \tag{B-43}$$

These may be formed into the matrix equation (B-28) .

$$\delta\dot{\underline{s}} = A\,\delta\underline{s} + \delta\underline{f}\,(t) \tag{B-44}$$

or

$$\dot{\delta\underline{s}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2p/cm^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \delta\underline{s} + \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{m}\,\delta(2p/c) \\ 0 \\ -\,\delta(2p/c^2) \end{Bmatrix} \qquad \text{(B-45)}$$

The column vector containing the power and exhaust velocity variations corresponds to the arbitrary forcing functions, $\underline{f}(t)$, of section B. 2 and to the product $B\,\delta\underline{f}$ in the general guidance problem. They may be written in any manner convenient for the investigation at hand. They are of no further concern at the moment and will be carried along as $B\,\delta\underline{f}$.

The adjoint functions may now be determined by direct integration of equation (B-46).

$$\dot{\Lambda} + \Lambda A = O_5 \qquad \text{(B-46)}$$

$$\Lambda(T) = I_5 \qquad \text{(B-47)}$$

Expanding the above and integrating the twenty-five equations one obtains

$$\Lambda = \begin{bmatrix} 1 & 0 & (t_f - t) & 0 & -\frac{c}{\dot{m}}\begin{Bmatrix} \left(1 - \frac{m_f}{m} + \ln\frac{m_f}{m}\right) & t > t_R \\ \left(1 - \frac{2m_f}{m_R} + \frac{m_f}{m} - \ln\frac{m_R^2}{mm_f}\right) & t < t_R \end{Bmatrix} \\ 0 & 1 & 0 & (t_f - t) & 0 \\ 0 & 0 & 1 & 0 & -c\begin{Bmatrix} \left(\frac{1}{m} - \frac{1}{m_f}\right) & t > t_R \\ \left(\frac{2}{m_R} - \frac{1}{m_f} - \frac{1}{m}\right) & t < t_R \end{Bmatrix} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(B-48)

Thus the adjoint set has been determined for a simple case and may be used in the study of guidance techniques.

APPENDIX C

PROPERTIES OF THE FUNDAMENTAL GUIDANCE EQUATION

C. 1 Summary

The contents of this appendix include discussions relative to interchanging the control vector and relative to properties of the adjoint set. In the absence of control perturbations the adjoint set alone describes the effect of state perturbations and thus may be called the "state transistion matrix". Several of its interesting properties are discussed.

Proof of singularities in the guidance equation is presented in section C. 5.

C. 2 General Remarks

To provide a better understanding of the methods used in the solution of the equations of motion, and to show the relationship between trajectory determination and guidance it is beneficial to examine various formulations of the variational equations of motion and the constraining equations.

A by-product of using the adjoint method with the calculus of variations to find an optimum trajectory is the solution of the adjoint set of equations. Physically the adjoint set represents "sensitivity" coefficients or "influence" functions which permit the investigator searching for a trajectory to determine the changes in trajectory parameters which will move the solution in the direction of the desired optimum. It is interesting to note that along the optimum trajectory the Lagrange multipliers are linear combinations of the adjoint variables. From the viewpoint of one searching for a trajectory, the adjoint set has served its purpose once the optimum trajectory is determined. However, for the guidance analyst the adjoint functions serve a most useful purpose by showing the effect of spacecraft perturbations on the final state of the vehicle.

C. 3 Interchanging the Control Vector

The equations of motion and the mass rate equations, expressed in a nonrotating frame with origin at the central body, are

$$\dot{\underline{r}} = \underline{v} \tag{C-1}$$

$$\dot{\underline{v}} = -\frac{\mu}{r^3}\,\underline{r} + \underline{a} \tag{C-2}$$

$$\dot{m} = g_m\,(m, \underline{a}) \tag{C-3}$$

or

$$\dot{m} = g_m\,(\underline{f}) \tag{C-4}$$

The variables $\dot{m}$ and $\underline{a}$ have a functional dependence which must be explicitly taken into account. In the subsequent discussion only the VSI case will be derived since CSI control follows an analogous argument.

The relationship between $\dot{m}$ and $\underline{a}$ may be written in any manner which is convenient to the argument at hand. If one is interested in the effect of acceleration changes irrespective of their cause then write

$$\dot{m} = -\frac{a^2\,m^2}{2p} \tag{C-5}$$

The analysis of Chapter II shows that equation (C-5) is the correct form for mass rate. By taking the first variations of equation (C-1) through (C-3) the state variational equation is obtained

$$\delta\dot{\underline{s}} = {}_a A\;\delta\underline{s} + {}_a B\;\delta\underline{a} \tag{C-6}$$

where

$$ {}_a A = \begin{bmatrix} O_3 & I_3 & \underline{O} \\ -G & O_3 & \underline{O} \\ \underline{O}^T & \underline{O}^T & \dfrac{-a^2 m}{p} \end{bmatrix} \tag{C-7}$$

$$ {}_a B = \begin{bmatrix} O_3 \\ I_3 \\ \dfrac{-m^2}{p}\,\underline{a}^T \end{bmatrix} \tag{C-8}$$

where the pre-subscript denotes that $\underline{a}$ is the control. The submatrices of A and B are the three by three null matrix O_3; the three by three identity I_3; the three by three symmetric matrix of gravitational gradients G; and the three by one null vector $\underline{O}$. If the adjoint set for equation (C-6) is formed such that

$$\dot{\Lambda} = -\Lambda A \tag{C-9}$$

$$\Lambda(t_f) = I \tag{C-10}$$

then the solution, Λ, must have the form

$$_a\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{O} \\ \Lambda_{21} & \Lambda_{22} & \underline{O} \\ \underline{O}^T & \underline{O}^T & \Lambda_{33} \end{bmatrix} \tag{C-11}$$

where the matrix is partitioned into three by three submatrices, three by one vectors and a scalar, Λ_{33}.

The form becomes apparent when equation (C-9) is expanded and an initial condition is applied. Notice also that for the form (C-11) to exist it is not critical that $\Lambda(t_f) = I$, only that at some time t, $\Lambda(t) = I$

The expansion of (6-9) yields

$$\dot{\Lambda}_{11} = \Lambda_{12} G \tag{C-12}$$

$$\dot{\Lambda}_{12} = -\Lambda_{11} \tag{C-13}$$

$$\dot{\underline{\Lambda}}_{13} = \underline{0} \tag{C-14}$$

$$\dot{\Lambda}_{21} = \Lambda_{22} G \tag{C-15}$$

$$\dot{\Lambda}_{22} = -\Lambda_{21} \tag{C-16}$$

$$\dot{\underline{\Lambda}}_{23} = \underline{0} \tag{C-17}$$

$$\dot{\underline{\Lambda}}_{31}^T = \underline{\Lambda}_{32}^T G \tag{C-18}$$

$$\dot{\underline{\Lambda}}_{32}^T = -\underline{\Lambda}_{31}^T \tag{C-19}$$

$$\dot{\Lambda}_{33} = \frac{a^2 m}{p} \Lambda_{33} \tag{C-20}$$

The form of equation (C-11) is assured if at some time $\Lambda = I$.

Now rewrite the functional dependence of $\underline{a}$ and $\dot{m}$ as

$$\underline{a} = \frac{\underline{f}}{m} \tag{C-21}$$

$$\dot{m} = -\frac{f^2}{2p} \tag{C-22}$$

where again f, m and p are per unit initial mass. The A and B matrix must be modified to

$$_fA = \begin{bmatrix} O_3 & I_3 & \underline{O} \\ -G & O_3 & -\frac{\underline{a}}{m} \\ \underline{O}^T & \underline{O}^T & 0 \end{bmatrix} \tag{C-23}$$

$$_fB = \begin{bmatrix} O_3 \\ \frac{1}{m} I_3 \\ \frac{-\underline{f}^T}{p} \end{bmatrix} \tag{C-24}$$

where the subscript denotes that $\underline{f}$ is the control. The state variational differential equation is

$$\delta\dot{\underline{s}} = {_fA}\,\delta\underline{s} + {_fB}\,\delta\underline{f} \tag{C-25}$$

Clearly the optimal trajectory must be independent of the particular control variation used in the computation scheme. That is, if the variational problem for computing trajectories is solved using $\underline{f}$ as the control, it must result in the same trajectory that would be obtained if $\underline{a}$ is used as the control. Thus, two different formulations are available which may be used at the discretion of the investigator. Since the cost has been specified as $a^2/2$ it is more convenient to use $\underline{a}$ as the control for computing trajectories. Nevertheless, the use of $\underline{f}$ as a control has important significance.

The adjoint set which corresponds to using $\underline{f}$ as the control will now be determined. Expanding (C-9) using (C-23) one obtains differences from the set (C-12) through (C-20) only in the last column of the solution Λ. That is

$$\dot{\underline{\Lambda}}_{13} = \Lambda_{12} \frac{a}{m} \tag{C-26}$$

$$\dot{\underline{\Lambda}}_{23} = \Lambda_{22} \frac{a}{m} \tag{C-27}$$

$$\dot{\Lambda}_{33} = 0 \tag{C-28}$$

For boundary conditions $\Lambda(t_f) = I$, the solution has the form

$$_f\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{\Lambda}_{13} \\ \Lambda_{21} & \Lambda_{22} & \underline{\Lambda}_{23} \\ \underline{O}^T & \underline{O}^T & 1 \end{bmatrix} \tag{C-29}$$

The solutions $_a\Lambda$ and $_f\Lambda$ differ only in the last column. The remaining numbers of the array are identical for each point in time.

The physical significance of this difference is apparent from the following physical reasoning. Consider the adjoint set as an array of partial derivatives. If $\Lambda(t_f) = I$ then the last column is

$$\begin{Bmatrix} \underline{\Lambda}_{13} \\ \underline{\Lambda}_{23} \\ \Lambda_{33} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \underline{r}(t_f)}{\partial m(t)} \\ \dfrac{\partial \underline{v}(t_f)}{\partial m(t)} \\ \dfrac{\partial m(t_f)}{\partial m(t)} \end{Bmatrix} \tag{C-30}$$

If $\underline{a}$ is the control then one should expect $\underline{\Lambda}_{13}$ and $\underline{\Lambda}_{23}$ to be null vectors, since for a given acceleration program variations in mass cannot affect the end point in position and velocity. However the end point for mass must change because it requires a different amount of propellant for different size vehicles.

If $\underline{f}$ is the control and the thrust program is given, then changes in mass will affect the end points in position and velocity due to the change in acceleration. For a given thrust program the final mass, however, can change only if the initial mass changes, therefore $\Lambda_{33} = 1$.

It is concluded that the control vectors $\delta\underline{f}$ and $\delta\underline{a}$ can be interchanged in the guidance equation simply by changing the last column of the adjoint set and changing the matrix B. Along the optimal trajectory all of these functions are known thus the interchange can be easily made at the discretion of the investigator.

The purpose of interchanging control vectors is simply stated. Computation of trajectories is most conveniently carried out with $\underline{a}$ as the control, due to the formulation of the cost. Assume however, that it is desired to investigate the effects of changes in engine performance or to investigate variables other than $\underline{a}$ as quantities to be measured. Then it is convenient to write, from Chapter II

$$\underline{f} = \dot{m}\,\underline{c} \tag{C-31}$$

then

$$\delta\underline{f} = \dot{m}\,\delta\underline{c} + \underline{c}\,\delta\dot{m} \tag{C-32}$$

or

$$\delta\underline{f} = [\underline{c}\ \ \dot{m} I_3] \begin{Bmatrix} \delta\dot{m} \\ \delta\underline{c} \end{Bmatrix} \tag{C-33}$$

Further, let

$$\underline{f} = \frac{-2p}{c^2}\,\underline{c} \tag{C-34}$$

then

$$\delta\underline{f} = -\frac{2\underline{c}}{c^2}\,\delta p - \frac{2p}{c^2}\left[I_3 - \frac{2\underline{c}\,\underline{c}^T}{c^2} \right] \delta\underline{c} \tag{C-35}$$

or

$$\delta\underline{f} = -\frac{2}{c^2}\left[\underline{c}\ \ \left[I_3 - \frac{2\underline{c}\,\underline{c}^T}{c^2} \right] \right] \begin{Bmatrix} \delta p \\ \delta\underline{c} \end{Bmatrix} \tag{C-36}$$

If in equation (C-25) or in the guidance equation, $\delta\underline{f}$ is replaced by either (C-33) or (C-36) it is possible to study the effects of the particular variations represented without computing a new adjoint set.

The ability to interchange control vectors at will, without requiring a complete new set of computations makes the fundamental guidance equation a very powerful tool.

C. 4 Properties of the State Transition Matrix

If the fundamental guidance equation is written for two different times $t = t_i$ and $t = t_j$ and subtracted, then

$$\Lambda(t_i)\,\delta \underline{s}_i = \Lambda(t_j)\,\delta \underline{s}_j + \int_{t_j}^{t_i} \Lambda B\,\delta \underline{a}\,dt \qquad \text{(C-37)}$$

or

$$\delta \underline{s}_i = T_{ij}\,\delta \underline{s}_j + \Lambda^{-1}(t_i) \int_{t_j}^{t_i} \Lambda B\,\delta \underline{a}\,dt \qquad \text{(C-38)}$$

where

$$T_{ij} = \Lambda^{-1}(t_i)\,\Lambda(t_j) \qquad \text{(C-39)}$$

In the absence of control perturbations, T_{ij} completely describes the transformation of the state vector at t_j into the state vector at t_i. Thus T_{ij} is called the "state transition matrix". It is necessary to show that T_{ij} is nonsingular. It is desirable to show that it can be inverted by inspection and that $T_{ij}^{-1} = T_{ji}$. This last property is readily observed by taking the inverse of equation (C-39).

$$(T_{ij})^{-1} = (\Lambda_i^{-1}\Lambda_j)^{-1} \qquad \text{(C-40)}$$

$$(T_{ij})^{-1} = T_{ji} = \Lambda_j^{-1}\Lambda_i \qquad \text{(C-41)}$$

To examine the other properties, it is necessary to look first at Λ. From section C. 3 write Λ in its general form

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{\Lambda}_{13} \\ \Lambda_{21} & \Lambda_{22} & \underline{\Lambda}_{23} \\ \underline{O}^T & \underline{O}^T & \Lambda_{33} \end{bmatrix} \qquad \text{(C-42)}$$

Equation (C-42) is to be understood as representing both the adjoint set with $\underline{f}$ as control and the set for $\underline{a}$ as control. The last column takes on the value appropriate to the control.

Now, partition Λ and consider only that portion defined by

$$H_j = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}_j \qquad \text{(C-43)}$$

where the subscript refers to time t_j. The differential equations for this subset of Λ are reproduced from section C. 3.

$$\dot{\Lambda}_{11} = \Lambda_{12}\, G \qquad \text{(C-12)}$$

$$\dot{\Lambda}_{12} = -\Lambda_{11} \qquad \text{(C-13)}$$

$$\dot{\Lambda}_{21} = \Lambda_{22}\, G \qquad \text{(C-15)}$$

$$\dot{\Lambda}_{22} = -\Lambda_{21} \qquad \text{(C-16)}$$

Observe that these equations are independent of the control and are completely specified by the physical path and the boundary conditions.

Using C-13 and C-16, H_j may be rewritten as

$$H_j = \begin{bmatrix} -\dot{\Lambda}_{12} & \Lambda_{12} \\ -\dot{\Lambda}_{22} & \Lambda_{22} \end{bmatrix}_j \qquad \text{(C-44)}$$

Now define a new matrix H_j^*.

$$H_j^* = \begin{bmatrix} \Lambda_{12} & \Lambda_{22} \\ \dot{\Lambda}_{12} & \dot{\Lambda}_{22} \end{bmatrix} \qquad \text{(C-45)}$$

H_j^* and H_j are related by

$$H_j^{*T} = H_j P \qquad \text{(C-46)}$$

where

$$P = \begin{bmatrix} O_3 & -I_3 \\ I_3 & O_3 \end{bmatrix} \qquad \text{(C-47)}$$

P is a skew symmetric matrix and possess the properties:

$$P^2 = -I \qquad \text{(C-48)}$$

$$P^T = P^{-1} = -P \qquad \text{(C-49)}$$

Since in equation (C-46) t_j is an arbitrary time along the trajectory, the subscript may be dropped and H^{*T} considered as a time varying matrix.

$$H^{*T} = H P \qquad (C\text{-}50)$$

Differentiating:

$$\dot{H}^{*T} = \dot{H} P \qquad (C\text{-}51)$$

From equations (C-12) through (C-16) and (C-44) it is possible to write

$$\dot{H} + H A^* = O_6 \qquad (C\text{-}52)$$

where

$$A^* = \begin{bmatrix} O_3 & I_3 \\ -G & O_3 \end{bmatrix} \qquad (C\text{-}53)$$

Substituting (C-52) into (C-51) and using (C-50) one obtains

$$\dot{H}^{*T} = + H^{*T} P A^* P \qquad (C\text{-}54)$$

Multiplying out the known product yields

$$P A^* P = A^{*T} \qquad (C\text{-}55)$$

since G is a symmetric matrix. Thus the transpose of equation (C-54) is

$$\dot{H}^* = A^* H^* \qquad (C\text{-}56)$$

Comparing (C-52) and (C-56) it is observed that H and H* satisfy the adjoint-fundamental relationship mentioned in Chapter III. That is

$$\frac{d}{dt} (H H^*) = O_6 \qquad (C\text{-}57)$$

or integrating

$$H_f^* = H_j H_j^* \qquad (C\text{-}58)$$

where

$$H(t_f) = H_f = I_6 \qquad (C\text{-}59)$$

Substituting equation (C-50) into (C-58) and using (C-49) obtain

$$-P H_f^T = - H_j P H_j^T \qquad (C\text{-}60)$$

Using (C-59)

$$H_j \, P H_j^T = P \tag{C-61}$$

Any matrix H exhibiting the property of equation (C-61) is said to be symplectic and its inverse is given by

$$H^{-1} = - P H^T P \tag{C-62}$$

The inverse of the submatrix H of Λ has been obtained by rearrangement of the components. To obtain Λ^{-1} two cases must be considered: the adjoint sets for the control $\underline{a}$ and for the control $\underline{f}$.

If $\Lambda = {}_a\Lambda$, then

$$_a\Lambda = \begin{bmatrix} H & \underline{O}_6 \\ \underline{O}_6^T & \Lambda_{33} \end{bmatrix} \tag{C-63}$$

where $\underline{O}_6$ is a 6 component null vector. Further

$$_a\Lambda^{-1} = \begin{bmatrix} H^{-1} & \underline{O}_6 \\ \underline{O}_6^T & \dfrac{1}{\Lambda_{33}} \end{bmatrix} \tag{C-64}$$

If $\Lambda = {}_f\Lambda$, then

$$_f\Lambda = \begin{bmatrix} H & \underline{\Lambda}_6 \\ \underline{O}_6^T & 1 \end{bmatrix} \tag{C-65}$$

where $\underline{\Lambda}_6$ denotes $\begin{Bmatrix} \underline{\Lambda}_{13} \\ \underline{\Lambda}_{23} \end{Bmatrix}$.

Equation (C-65) may be written as the product of two matrices.

$$_f\Lambda = \begin{bmatrix} -I_6 & \underline{\Lambda}_6 \\ \underline{O}_6^T & 1 \end{bmatrix} \begin{bmatrix} -H & \underline{O}_6 \\ \underline{O}_6^T & 1 \end{bmatrix} \tag{C-66}$$

Then

$$_f\Lambda^{-1} = \begin{bmatrix} -H^{-1} & \underline{O}_6 \\ \underline{O}_6^T & 1 \end{bmatrix} \begin{bmatrix} -I & \underline{\Lambda}_6 \\ \underline{O}_6^T & 1 \end{bmatrix} \tag{C-67}$$

$$
{}_f\Lambda^{-1} = \begin{bmatrix} H^{-1} & -H^{-1}\underline{\Lambda}_6 \\ \underline{O}_6^T & 1 \end{bmatrix} \tag{C-68}
$$

since the second term in (C-67) is its own inverse. Therefore Λ has now been inverted for all cases. Now consider T_{ij}.

$$
T_{ij} = \Lambda_i^{-1} \Lambda_j \tag{C-39}
$$

For $\Lambda = {}_a\Lambda$, use (C-63) and (C-64)

$$
{}_aT_{ij} = \left[\begin{array}{c|c} H_i^{-1} H_j & \underline{O}_6 \\ \hline \underline{O}_6^T & \dfrac{\Lambda_{33\,j}}{\Lambda_{33\,i}} \end{array}\right] \tag{C-69}
$$

For $\Lambda = {}_f\Lambda$, use (C-65) and (C-68)

$$
{}_fT_{ij} = \left[\begin{array}{c|c} H_i^{-1} H_j & -H_i^{-1}\left\{\underline{\Lambda}_{6i} - \underline{\Lambda}_{6j}\right\} \\ \hline \underline{O}_6^T & 1 \end{array}\right] \tag{C-70}
$$

Finally it is necessary to show that T_{ij} is nonsingular. To do this it is sufficient to show that the determinant of $H_i^{-1} H_j$ is never zero and that Λ_{33} is never zero.

From equation (C-20) it is apparent that if $\Lambda_{33}(0) = 1$ it increases monotonically, thus is never zero.

Now examine the determinant of

$$
H_i^{-1} H_j
$$

From (C-61)

$$
\det H P H^T = \det P = 1 \tag{C-71}
$$

Since the determinant of a product is the product of the determinants

$$
(\det H)^2 = 1 \tag{C-72}
$$

From equation (C-62)

$$\det H^{-1} = \det(-P H^T P) \tag{C-73}$$

$$\det H^{-1} = \det H \tag{C-74}$$

$$\text{since } \det(-P) = (-1)^k \det P = 1 \tag{C-75}$$

where k = 6 is the order of P.

Since the subscripts i and j are completely arbitrary and since H is continuous,

$$\det H_i^{-1} H_j = \det H^{-1} H = 1 \tag{C-76}$$

Thus T_{ij} is nonsingular.

C. 5 Singularities of the Guidance Equation

Solution of the guidance equation for a control program which will satisfy certain terminal conditions invariably results in the inversion of a matrix integral expression. In section 3. 8 the solution for FTA guidance requires such an inversion, namely

$$M^{-1} = \left[\int_{t_1}^{t_f} \Lambda^* B B^T \Lambda^{*T} dt \right]^{-1} \tag{C-77}$$

In this section the proof will be given that M is not singular for $(t_f - t_1) > 0$ but that the corresponding matrix denoted by M_o, with Λ in place of Λ^*, is singular in the case of unconstrained control programs, for all values of t_1 along the optimal trajectory. We shall proceed to prove that M_o is singular and by induction show that M is not singular.

The following proof of the singularity in M_o was suggested by Dr. James Potter.

If M_o is singular then there is a nonzero vector $\underline{p}_o$ such that

$$M_o \underline{p}_o = 0 \tag{C-78}$$

M_o is symmetric thus it is also true that

$$\underline{p}_o^T\ M_o = 0 \qquad \text{(C-79)}$$

and

$$\underline{p}_o^T\ M_o \underline{p}_o = 0 \qquad \text{(C-80)}$$

This implies that

$$0 = \int_{t_1}^{t_f} \underline{p}_o^T\ \Lambda B B^T\ \Lambda^T \underline{p}_o\ dt \qquad \text{(C-81)}$$

However equation (C-81) is the integral of the square of the length of a vector

$$\int_{t_1}^{t_f} \left| B^T\ \Lambda^T\ \underline{p}_o \right|^2 dt = 0 \qquad \text{(C-82)}$$

Thus for M_o to be singular it is necessary and sufficient that the vector integrand of (C-82) be zero.

$$B^T(t)\ \Lambda^T(t)\ \underline{p}_o = 0 \qquad \text{(C-83)}$$

for all $t_1 < t < t_f$.

That a vector $\underline{p}_o$ exists for the optimum trajectory such that equation (C-83) is satisfied may be shown by application of the Pontryagin maximum principle for unconstrained control. The Pontryagin maximum principle asserts that if the final value, S, of some combination of the state variables, $\underline{x}$, are to be a maximum or minimum with respect to the control variables, $\underline{y}$; that is,

$$\min_{y_i} S = \underline{d}^T\ \underline{x}_f \qquad \text{(C-84)}$$

then the desired path will be one such that the gradient of the state velocity, $\dot{\underline{x}}$, in the vector space of all admissible $\underline{y}$,

$$\dot{\underline{x}} = \underline{g}\ (\underline{x},\ \underline{y},\ t) \qquad \text{(C-85)}$$

will be orthogonal to a vector $\underline{p}$ which satisfies the differential equations adjoint to $\dot{\underline{x}} = \underline{g}(\underline{x}, \underline{y}, t)$ and such that the final value of $\underline{p}$ is

$$\underline{p}_f = -\underline{d} + \underline{\nu}_f \tag{C-86}$$

where $d_i = 0$ for components of the state vector which are fixed at t_f and $\nu_{i_f} = 0$ for components of the state vector to be minimized or maximized.

Thus to satisfy equation (C-84), it is necessary and sufficient that

$$\underline{p}^T \left[\frac{\partial (\underline{g}(\underline{x}, \underline{y}, t))}{\partial \underline{y}} \right] = 0 \tag{C-87}$$

where

$$\dot{\underline{x}} = \underline{g}(\underline{x}, \underline{y}, t) \tag{C-85}$$

$$\dot{\underline{p}} = -\left[\frac{\partial (\underline{g}(\underline{x}, \underline{y}, t))}{\partial \underline{x}} \right]^T \underline{p} \tag{C-88}$$

$$\underline{p}_f = -\underline{d} + \underline{\nu}_f \tag{C-86}$$

A proof of the maximum principle is given in Appendix D.

Application of equations (C-86), (C-87), (C-88) to the guidance problem in this thesis reveals that

$$\underline{p}_f = -\begin{Bmatrix} \underline{O}_6 \\ 1 \end{Bmatrix} + \begin{Bmatrix} \underline{\nu}_6 \\ 0 \end{Bmatrix}_f = \begin{Bmatrix} \underline{\nu}_6 \\ -1 \end{Bmatrix} \tag{C-89}$$

$$\dot{\underline{p}} = -A^T \underline{p} \tag{C-90}$$

$$\underline{p}^T B = 0 \tag{C-91}$$

where A^T and B are the matrices of partial derivatives in (C-88) and (C-87) respectively.

Since $\underline{p}$ satisfies equation (C-90) and also Λ^T satisfies (C-90), a solution of $\underline{p}$ is

$$\underline{p} = \Lambda^T \underline{p}_f \tag{C-92}$$

Then substituting into equation (C-91)

$$\underline{p}_f^T \Lambda B = 0 \tag{C-93}$$

for all $t_1 < t < t_f$. Thus $\underline{p}_f$ is a null vector of M_o. Furthermore, since t_1 is arbitrary

$$M_o(t_1)\, \underline{p}_f = 0 \qquad (0 < t_1 < t_f) \tag{C-94}$$

To show that M is not singular it is sufficient to show that every null vector of M_o satisfies the boundary condition on $\underline{p}$, equation (C-86). This sufficiency condition is easily verified by noting that the final element of $\underline{p}_f$ is invariant and that M may be obtained from M_o by deleting the last row and last column of M_o. In general, if Λ is interpreted as the nonsingular transformation that transforms $\underline{p}_f^T$ into $\underline{p}^T$ and if $\underline{p}^T$ is a null vector of B because of the component p_{if}, then deleting the ith row of Λ to produce Λ^* destroys the transformation which carries p_{if} into $\underline{p}$.

The fact that only the last row of Λ need be deleted is a consequence of the fact that only the last component of the final state vector, m_f, has been maximized in obtaining the optimum path. Therefore, we assert that if every null vector of M_o satisfies equation (C-89), the M obtained in equation (C-77) is nonsingular.

Let us prove then that all null vectors of M_o satisfy the boundary condition. For the problem in this thesis B is a seven by three matrix of rank three and contains zeros in the first three rows. By virtue of the rank, the three columns of B are linearly independent. Further, since B is of rank three there are four linearly independent vectors which satisfy (C-91) and thus are null vectors of B. Three of the four vectors may be chosen to satisfy (C-91) by virtue of the three rows of zeros in B. These vectors do not satisfy (C-92) and (C-93) and therefore are not null vectors of M_o. The fourth vector must be a null vector of M_o since the three columns of B and the four null vectors of B form a basis in seven dimensional vector space and six of the vectors are not null vectors of M_o. This vector is unique and must satisfy (C-89) through (C-93).

The proof is thus complete since all vectors may be written as linear combinations of the vectors in the basis.

APPENDIX D

PONTRYAGIN'S MAXIMUM PRINCIPLE

D. 1 Introduction

In this appendix a derivation of Pontryagin's maximum principle is given which is applicable to the problem of the thesis.

Consider that the cost function S is some scalar function of the final state variable, $\underline{x}_f$. It is always possible to cast a problem into this form by redefining variables. The control vector will be denoted by a generalized vector $\underline{u}$, which is bounded by certain constraints. In addition, the solution must satisfy boundary conditions on some of the state variables. The desired solution is the optimal control, $\underline{u}^o$, which will maximize the cost and satisfy all constraining conditions.

D. 2 The Maximum Principle

Assume that the state variables may be written in the form

$$\dot{\underline{x}} = \underline{g}\,(\underline{x}, \underline{u}) \tag{D-1}$$

and that the solution is to satisfy the final boundary conditions

$$\underline{\phi}\,(\underline{x}_f) = \underline{0} \tag{D-2}$$

The cost function may be written as

$$S = \underline{d}^T \underline{x}_f + \underline{\nu}^T \underline{\phi} \tag{D-3}$$

where $\underline{d}$ is a known vector and $\underline{\nu}$ is an unknown constant vector.

Then small changes in the cost due to changes in state are given by

$$\delta S = \underline{d}^T \delta \underline{x}_f + \underline{\nu}^T \frac{\partial \underline{\phi}}{\partial \underline{x}_f} \delta \underline{x}_f \tag{D-4}$$

where

$$\frac{\partial \phi_i}{\partial x_{if}} = 1 \text{ or } 0 \tag{D-5}$$

Equation (D-5) is 1 for fixed boundaries; 0 for unspecified boundaries.

The variables adjoint to (D-1) are given by

$$\dot{\underline{p}} = - A^T \underline{p} = - \left(\frac{\partial \underline{g}}{\partial \underline{x}} \right)^T \underline{p} \tag{D-6}$$

$$\underline{p}\,(t_f) = \underline{p}_f \tag{D-7}$$

The vector $\underline{p}$ is frequently called the "costate".

Assume that an optimal control, $\underline{u}^o$, is known. Then for small changes $\delta\underline{u}$, the state must change by $\delta\underline{x}$. Then

$$\delta\dot{\underline{x}} = \underline{g}\,(\underline{x}^o + \delta\underline{x}, \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o, \underline{u}^o) \tag{D-8}$$

Consider the term

$$\frac{d}{dt}(\underline{p}^T \delta\underline{x}) = \underline{p}^T \delta\dot{\underline{x}} + \dot{\underline{p}}^T \delta\underline{x} \tag{D-9}$$

Applying (D-8) one obtains

$$\dot{\underline{p}}^T \delta\underline{x} = \frac{d}{dt}(\underline{p}^T \delta\underline{x}) - \underline{p}^T [\underline{g}\,(\underline{x}^o + \delta\underline{x}, \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o, \underline{u}^o)] \tag{D-10}$$

Integrating (D-10) yields

$$\int_0^{t_f} \dot{\underline{p}}^T \delta\underline{x}\, dt = \underline{p}^T \delta\underline{x} \Big|_0^{t_f} - \int_0^{t_f} \underline{p}^T [\underline{g}\,(\underline{x}^o + \delta\underline{x}, \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o, \underline{u}^o)]\, dt \tag{D-11}$$

If the initial state boundary is fixed, $\delta\underline{x}_o = 0$ and the first right hand term is

$$\delta S = \underline{p}_f^T \, \delta\underline{x}_f \tag{D-12}$$

provided $\underline{p}_f$ is chosen such that

$$\underline{p}_f^T = \underline{d}^T + \underline{\nu}^T \frac{\partial \underline{\phi}}{\partial \underline{x}_f} \tag{D-13}$$

The right side integrand may be expanded in a Taylor series around the optimal trajectory if the variations due to control and due to state are

separated. That is

$$\underline{g}\,(\underline{x}^o + \delta\underline{x},\ \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o,\ \underline{u}^o) = \underline{g}\,(\underline{x}^o,\ \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o,\ \underline{u}^o)$$

$$+ \frac{\partial}{\partial \underline{x}}[\underline{g}\,(\underline{x}^o,\ \underline{u}^o + \delta\underline{u})]\,\delta\underline{x} + \text{remainder} \tag{D-14}$$

If the adjoint relationship, (D-6), is used on the left hand integrand of (D-11), the term may be rewritten as

$$\int_0^{t_f} \dot{\underline{p}}^T\,\delta\underline{x}\,dt = -\int_0^{t_f} \underline{p}^T\,\frac{\partial}{\partial \underline{x}}[\underline{g}\,(\underline{x}^o,\ \underline{u}^o)]\,\delta\underline{x}\,dt \tag{D-15}$$

Then substituting (D-12), (D-14) and (D-15) into (D-11), the result is

$$\delta S = \int_0^{t_f} \underline{p}^T[\underline{g}\,(\underline{x}^o,\ \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o,\ \underline{u}^o)]\,dt$$

$$+ \int_0^{t_f} \underline{p}^T\left\{\frac{\partial}{\partial \underline{x}}[\underline{g}\,(\underline{x}^o,\ \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o,\ \underline{u}^o)]\right\}\delta\underline{x}\,dt \tag{D-16}$$

Consider the first integrand in (D-16). If S is a maximum, then any allowable change $\delta\underline{u}$ must cause δS to be negative. Therefore for all $0 \leq t \leq t_f$

$$\underline{p}^T\,[\underline{g}\,(\underline{x}^o,\ \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o,\ \underline{u}^o)] < 0 \tag{D-17}$$

In considering the second integrand it is argued that admissible control changes $\delta\underline{u}$, can produce only a small variation in the derivative term such that its product with $\delta\underline{x}$ is second order or higher. Consequently the second integral and all higher order remainder terms of (D-14) are neglected.

Therefore (D-17) represents the sufficient condition for an optimum. A necessary condition is

$$\underline{p}^T\,[\underline{g}\,(\underline{x}^o,\ \underline{u}^o + \delta\underline{u}) - \underline{g}\,(\underline{x}^o,\ \underline{u}^o)] \leq 0 \tag{D-18}$$

The implication of (D-18) is that for all points on the optimal trajectory and for all admissible functions $\underline{u}$, $\underline{p}^T\underline{g}$ must be a maximum. Replacing $\underline{g}$ with $\dot{\underline{x}}$, notice that the scalar product of $\underline{p}$ and the state velocity must be a maximum.

If the control is unconstrained, then a necessary condition that $\underline{p}^T \underline{g}$ is a maximum is that

$$\frac{\partial}{\partial \underline{u}} (\underline{p}^T \underline{g}) = 0 = \underline{p}^T \frac{\partial \underline{g}}{\partial \underline{u}} \tag{D-19}$$

From previous work

$$B = \frac{\partial \underline{g}}{\partial \underline{u}} \tag{D-20}$$

Thus a necessary condition for a maximum is

$$\underline{p}^T B = \underline{O} \tag{D-21}$$

It is possible to derive the Hamiltonian formulation used in Chapter III by introducing a slightly different cost function S'. In place of (D-3) let

$$S' = \int_0^{t_f} h(\underline{u})\ dt + \underline{\nu}^T \underline{\phi} \tag{D-22}$$

where $h(\underline{u})$ is some function of the control. Then

$$\delta S' = \int_0^{t_f} \frac{\partial h}{\partial \underline{u}} \delta\underline{u}\ dt + \underline{\nu}^T \frac{\partial \underline{\phi}}{\partial \underline{x}_f} \delta\underline{x}_f \tag{D-23}$$

where again

$$\frac{\partial \phi_i}{\partial x_{if}} = 1 \text{ or } 0 \tag{D-5}$$

Equations (D-6) through (D-11) hold and (D-23) may be written as

$$\delta S' = \int_0^{t_f} [h(\underline{u}^o + \delta\underline{u}) - h(\underline{u}^o)]\ dt + \underline{p}_f^T \delta\underline{x}_f \tag{D-24}$$

provided that $\underline{p}_f$ is chosen such that

$$\underline{p}_f^T = \underline{\nu}^T \frac{\partial \underline{\phi}}{\partial \underline{x}_f} \tag{D-25}$$

If D-24) is substituted into (D-11) along with (D-14) and (D-15) , the result is

$$\delta S' = \int_{0}^{t_f} \left\{ \underline{p}^T [\underline{g}(\underline{x}^o, \underline{u}^o + \delta\underline{u}) - \underline{g}(\underline{x}^o, \underline{u}^o)] + [h(\underline{u}^o + \delta\underline{u}) - h(\underline{u}^o)] \right\} dt + \text{higher order terms} \tag{D-26}$$

Equation (D-26) is analagous to (D-16). A sufficient condition that S' be a maximum is that for all admissible control changes, $\delta\underline{u}$

$$\underline{p}^T [\underline{g}(\underline{x}^o, \underline{u}^o + \delta\underline{u}) - \underline{g}(\underline{x}^o, \underline{u}^o)] + [h(\underline{u}^o + \delta\underline{u}) - h(\underline{u}^o)] < 0 \tag{D-27}$$

The implication of (D-27) is that for all points on the optimal trajectory and for all admissible functions $\underline{u}^o$, $(\underline{p}^T \underline{g} + h(\underline{u}))$ must be a maximum. Using the VSI vehicle as an example

$$h(\underline{a}) = \frac{a^2}{2} \tag{D-28}$$

Then using (D-1) define

$$H = \frac{a^2}{2} + \underline{p}^T \underline{x} \tag{D-29}$$

For the linearized guidance problem of Chapter III, the vector $\dot{\underline{\xi}}$ was interpreted as the velocity of the state error at the final time. Since $\dot{\underline{\xi}}$ is a state velocity and $\underline{p}_f$ is the final value of the costate, using (D-25)

$$H = \frac{a^2}{2} + \underline{\nu}^T \dot{\underline{\xi}} \tag{D-30}$$

where for fixed boundary conditions

$$\frac{\partial \underline{\phi}}{\partial \underline{\xi}} = I \qquad \text{(D-31}$$

Equation (D-30) is a linearized version of the general Hamiltonian formulation, but with a nonlinear cost.

D. 3 Optimization Criterion

It was briefly mentioned in Chapter V that the optimality criterion for Pontryagin's maximum principle is more workable than the calculus

of variations criterion. This remark deserves further discussion. From the previous section it may be noted that the criterion for optimality may be stated: "For admissible control variations, the change in the cost function must be zero or negative (positive) if the cost is maximum (minimum). Admissible control variations are those which violate neither the boundary conditions nor any control constraint."

In contrast to this, the calculus of variations criterion for optimality is derived from the integral of the functional F used in Chapter V. Referring to equation (5-18), for example, the optimality criterion may be stated: "For admissible control variations and variations in state, the boundary conditions must be satisfied and the cost must not change to first order."

Since the integral of the functional equals the cost, the form of equation (5-18), rewritten here as (D-32),

$$\delta \int F\, dt = 0 = \Big[\quad \Big]_0^{t_f} + \int_0^{t_f} \Big[(\quad)\, \delta \underline{r} + (\quad)\, \delta \underline{v} + (\quad)\, \delta m + (\quad)\, \delta \underline{a} \Big]\, dt \tag{D-32}$$

gives the impression that the criterion must hold for all arbitrary variations in state and control. The implication is not true for the problem of this thesis and is misleading at best because (D-32) does not represent the first variation of the cost. It will now be shown that although (D-32) is satisfied for all variations in state and control, the optimality criterion can not be satisfied for all arbitrary state variations.

Let the guidance equation (3-27) be partitioned such that it consists of two equations

$$\begin{Bmatrix} \delta\, \underline{r} \\ \delta\, \underline{v} \end{Bmatrix}_f = \Lambda_t^* \, \delta \underline{s}_t + \int_t^{t_f} \Lambda^* B\, \delta \underline{a} \tag{D-33}$$

$$\delta m_f = \underline{\Lambda}_7^T \, \delta \underline{s}_t + \int_t^{t_f} \underline{\Lambda}_7^T B\, \delta \underline{a}\, dt \tag{D-34}$$

Using the expansion of the matrix Λ, (D-33) and (D-34) are

$$\begin{Bmatrix} \delta \underline{r} \\ \delta \underline{v} \end{Bmatrix}_f = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}_t \begin{Bmatrix} \delta \underline{r} \\ \delta \underline{v} \end{Bmatrix}_t + \int_t^{t_f} \Lambda^* B \, \delta \underline{a} \, dt \tag{D-35}$$

$$\delta m_f = \Lambda_{33} \, \delta m_t - \int_t^{t_f} \Lambda_{33} \left(\frac{m^2}{p} \right) \underline{a}^T \delta \underline{a} \, dt \tag{D-36}$$

Using the solution $\Lambda_{33} = m_f^2/m^2$ from equation (5-25), (D-36) may be reduced further.

$$\delta m_f = \frac{m_f^2}{m_t^2} \delta m_t - \frac{m_f^2}{p} \int_t^{t_f} \underline{a}^T \delta \underline{a} \, dt \tag{D-37}$$

For the criterion to be satisfied it is necessary that the final state variation $\delta \underline{s}_f$ vanish. First allow $\begin{Bmatrix} \delta \underline{r} \\ \delta \underline{v} \end{Bmatrix}$ to take on a nonzero value but let δm_t equal zero. Then (D-35) and (D-37) become

$$\underline{O} = \underline{\xi} + \int_t^{t_f} \Lambda^* B \, \delta \underline{a} \, dt \tag{D-38}$$

$$0 = 0 - \frac{m_f^2}{p} \int_t^{t_f} \underline{a}^T \delta \underline{a} \, dt \tag{D-39}$$

From Chapter III it is known that all propellant optimal control variations which satisfy the boundary conditions are

$$\delta \underline{a} = B^T \Lambda^{*T} M^{-1} (\underline{\eta} - \underline{\xi}) - \underline{a} \tag{D-40}$$

Substituting this solution into (D-39) and expanding yields

$$0 = -\frac{m_f^2}{p} \left[\underline{\eta}^T M^{-1} \underline{\eta} - 2J - \underline{\eta}^T M^{-1} \underline{\xi} \right] \tag{D-41}$$

However, along an optimal trajectory

$$\underline{\eta}^T M^{-1} \underline{\eta} - 2J = 0 \tag{D-42}$$

Thus to satisfy (D-39) it is necessary that

$$0 = \underline{\eta}^T M^{-1} \underline{\xi} \tag{D-43}$$

(D-43) does not hold in general. It holds only when the error $\underline{\xi}$ is such that a control change $\delta\underline{a}$ orthogonal to $\underline{a}$ can satisfy boundary conditions.

Now consider a variation in δm_f but no variation in position or velocity. Then it is required that

$$\underline{O} = \int_t^{t_f} \Lambda^* B \, \delta\underline{a} \, dt \tag{D-44}$$

$$0 = \frac{m_f^2}{m_t^2} - \frac{m_f^2}{p} \int_0^t \underline{a}^T \delta\underline{a} \tag{D-45}$$

A general variation $\delta\underline{a}$ may always be written as

$$\delta\underline{a} = B^T \Lambda^{*T} \underline{\pi} - \underline{a} \tag{D-46}$$

Inserting (D-46) into (D-45) and (D-44) and solving, one obtains from (D-44)

$$\underline{\pi} = M^{-1} \underline{\eta} \tag{D-47}$$

and from (D-45)

$$p \frac{\delta m_t}{m_t^2} = \underline{\eta}^T \underline{\pi} - 2J \tag{D-48}$$

Eliminating $\underline{\pi}$

$$p \frac{\delta m_t}{m_f^2} = \underline{\eta}^T M^{-1} \underline{\eta} - 2J = 0 \tag{D-49}$$

Therefore there are no variations δm_t which permit the boundary conditions to be satisfied and which do not change the cost.

It is concluded that the calculus of variation criterion and formulation can be misleading if not treated with much care. The Pontryagin principle avoids the difficulty by making no explicit requirements on state variations.

APPENDIX E

STEEPEST ASCENT FORMULATION FOR GUIDANCE

E. 1 Introduction

In this appendix the guidance problem will be formulated as it would appear in a "steepest ascent" trajectory computation procedure. The steepest ascent method is characterized by a rigid control over the step size between iterations. This step control is to prevent the procedure from violating linearity assumptions as it approaches the optimum. Sophisticated techniques have been devised for automatic selection of step size.[26] However, at some point the investigator must use his experience and judgement to select a constant or matrix of constants to insert in the selection procedure for step size. The author is convinced that the additional complexity of rigid step control cannot always be justified.

E. 2 Steepest Ascent Solution

For simplicity in illustrating the steepest ascent method, consider only the trajectory problem for the unrestricted VSI mode of control. In this problem it is desired that the optimal acceleration integral be a minimum.

$$J^o = \int_0^{t_f} \frac{|\underline{a}^o|^2}{2} \, dt \qquad \text{(E-1)}$$

It is further desired that the change in the acceleration program from one iteration to the next be such that

$$k^2 = \int_0^{t_f} |\delta \underline{a}|^2 \, dt \qquad \text{(E-2)}$$

where k^2 is a constant to be selected at the discretion of the investigator.

In order to satisfy, to first order, the terminal constraint $\underline{\xi}$, $\delta\underline{a}$ must satisfy the relation

$$- \underline{\xi} = \int_0^{t_f} \Lambda^* B \, \delta \underline{a} \, dt \tag{E-3}$$

where all variables are defined exactly as in Chapter III.

Define the new cost function J' such that

$$J' = \int_0^{t_f} \frac{\underline{a}^{oT} \underline{a}^o}{2} dt + \frac{\pi_1}{2}\left(k^2 - \int_0^{t_f} \delta \underline{a}^T \delta \underline{a} \, dt\right) + \underline{\pi}_2^T \left(\underline{\xi} + \int_0^{t_f} \Lambda^* B \, \delta \underline{a} \, dt\right) \tag{E-4}$$

where π_1 and $\underline{\pi}_2$ are constant Lagrange multipliers, a scalar and a vector respectively.

For arbitrary variations in $\delta \underline{a}$, the first variation of J' must vanish. Thus for

$$\underline{a}^o = \underline{a} + \delta \underline{a} \tag{E-5}$$

where $\underline{a}$ is a nonoptimal program,

$$0 = \left[(\underline{a} + \delta \underline{a})^T = \pi_1 \, \delta \underline{a}^T + \underline{\pi}_2^T \Lambda^* B \right] \delta (\delta \underline{a}) \tag{E-6}$$

$$\delta (\delta \underline{a}) \neq 0$$

or

$$\delta \underline{a} = \frac{-B^T \Lambda^{*T} \underline{\pi}_2 - \underline{a}}{(1 - \pi_1)} \tag{E-7}$$

The multipliers may be determined by satisfying the constraint equations. First solve for $\underline{\pi}_2$ through equation (E-3).

$$\underline{\pi}_2 = M^{-1} [(1 - \pi_1) \underline{\xi} - \underline{\eta}] \tag{E-8}$$

where

$$M = \int_0^{t_f} \Lambda^* B B^T \Lambda^{*T} dt \tag{E-9}$$

$$\underline{\eta} = \int_0^{t_f} \Lambda^* B \underline{a} \, dt \tag{E-10}$$

and

$$\delta \underline{a} = - B^T \Lambda^{*T} M^{-1} \left[\underline{\xi} - \frac{\underline{\eta}}{(1-\pi_1)} \right] - \frac{\underline{a}}{(1-\pi_1)} \tag{E-11}$$

Using (E-2) to solve for $\dfrac{1}{(1-\pi_1)}$ one obtains

$$(1-\pi_1)^2 = \frac{2J - \underline{\eta}^T M^{-1} \underline{\eta}}{k^2 - \underline{\xi}^T M^{-1} \underline{\xi}} \tag{E-12}$$

or

$$\frac{1}{(1-\pi_1)} = \sqrt{\frac{k^2 - \underline{\xi}^T M^{-1} \underline{\xi}}{2J - \underline{\eta}^T M^{-1} \underline{\eta}}} \tag{E-13}$$

where J is the acceleration integral for the nonoptimum program.

The factor $\left(\dfrac{1}{1-\pi_1}\right)$ is the step size control which must be determined by judicious selection of k^2. Observe, however, the result of using (E-11) to evaluate the integral of $\underline{a}^T \delta \underline{a}$. This term is the cross product in the expansion of $|\underline{a}^o|^2$ by equation (E-5).

$$\int_0^{t_f} \underline{a}^T \delta \underline{a} \, dt = - \underline{\eta}^T M^{-1} \underline{\xi} - \frac{1}{1-\pi_1} (2J - \underline{\eta}^T M^{-1} \underline{\eta}) \tag{E-14}$$

As the program $\underline{a}$ approaches $\underline{a}^o$, the difference, $\delta \underline{a}$, must vanish. Likewise the error $\underline{\xi}$ must vanish. From (E-14) either the step size must approach zero or the term $(2J - \underline{\eta}^T M^{-1} \underline{\eta})$ must approach zero or both. It is easy to show that both must approach zero. Consider

$$\underline{\alpha} \equiv B^T \Lambda^{*T} M^{-1} \underline{\eta} - \underline{a} \tag{E-15}$$

Then

$$\int_0^{t_f} \underline{\alpha}^T \underline{\alpha} \, dt = 2J - \underline{\eta}^T M^{-1} \underline{\eta} \tag{E-16}$$

Clearly

$$\int_0^{t_f} \underline{a}^T \underline{a} \, dt \geq 0 \qquad \text{(E-17)}$$

with equality holding only when

$$\underline{a} = B^T \Lambda^{*T} M^{-1} \underline{\eta} \qquad \text{(E-18)}$$

Equation (E-18) holds only along the optimal trajectory. Therefore from (E-13) it is clear that for $\left(\frac{1}{1-\pi_1}\right)$ to remain finite, $(k^2 - \underline{\xi}^T M^{-1} \underline{\xi})$ must approach zero at least as fast as $(2J - \underline{\eta}^T M^{-1} \underline{\eta})$. Since $|\underline{\xi}|$ approaches zero this implies that k^2 must be reduced to zero as the optimal path is approached. One concludes that "steepest ascent", as formulated using the usual technique [26], inherently converges slowly.

The approach in this thesis is to assume that as $|\dot{\underline{\xi}}|$ becomes small the step size will automatically become small without the arbitrary constraint imposed by k^2. Thus π_1 is set equal to zero and the procedure allowed to converge as rapidly as possible. The results appear to justify this procedure.

APPENDIX F

COMPUTATIONAL COORDINATES

F. 1 Introduction

In this appendix the computational coordinate system is derived from the ephemeris data of the launch planet and target planet. In the chapters describing the guidance equation and its uses, generalized vector notation is used and the problem of coordinates does not arise. The motion is assumed to be described in a nonrotating frame.

For computations, however, it is necessary to be more specific. The computational frame used in computer tests is a heliocentric frame defined by the transfer plane and the initial point. The transfer plane is the plane which contains the initial point, the final point and the sun. The x axis of the system passes through the initial point, the z axis is northerly and the y axis completes the triad.

The objective is to describe the transfer plane and its coordinate system in terms of ephemeris data. This data may be given in either ecliptic or equatorial coordinates. The description of these systems and the transformation between them is available in most basic celestial mechanics texts and will not be reproduced here.

F. 2 Computational Coordinates

Designate the unit vectors in any system as $\underline{i}$, $\underline{j}$, and $\underline{k}$ and attach subscripts to denote the system. For the computational system use subscript c. Then, if the launch and target points are denoted by subscript L and T respectively,

$$\underline{i}_c = \frac{\underline{r}_L}{r_L} \tag{F-1}$$

$$\underline{k}_c = \frac{\underline{r}_L \times \underline{r}_T}{\left|\underline{r}_L \times \underline{r}_T\right|} \tag{F-2}$$

$$\underline{j}_c = \underline{k}_c \times \underline{i}_c \tag{F-3}$$

The transfer angle, θ, in the transfer plane is determined by

$$\underline{k}_c \sin\theta = \frac{\underline{r}_L \times \underline{r}_T}{r_L r_T} \qquad \text{(F-4)}$$

$$\left|\underline{k}_c \sin\theta\right| = \sin\theta \qquad \text{(F-5)}$$

The coordinates of the launch (initial) point are

$$\begin{Bmatrix} x_L \\ y_L \\ z_L \end{Bmatrix} = \begin{Bmatrix} r_L \\ 0 \\ 0 \end{Bmatrix}_c \qquad \text{(F-6)}$$

The coordinates of the target point are

$$\begin{Bmatrix} x_T \\ y_T \\ z_T \end{Bmatrix} = \begin{Bmatrix} r_T \cos\theta \\ r_T \sin\theta \\ 0 \end{Bmatrix}_c \qquad \text{(F-7)}$$

Two additional parameters of interest are the inclination of the transfer plane to the planes of the launch planet and the target planet. If the northerly normals to these planes are denoted by $\underline{k}_L$ and $\underline{k}_T$ respectively, and the angles by a_L and a_T respectively, then

$$\cos a_L = \underline{k}_L \cdot \underline{k}_c \qquad \text{(F-8)}$$

$$\cos a_T = \underline{k}_T \cdot \underline{k}_c \qquad \text{(F-9)}$$

where

$$\underline{k}_L = \frac{\underline{r}_L \times \underline{r}_{(L+90^\circ)}}{\left|\underline{r}_L \times \underline{r}_{(L+90^\circ)}\right|} \qquad \text{(F-10)}$$

$$\underline{k}_T = \frac{\underline{r}_T \times \underline{r}_{(T+90^\circ)}}{\left|\underline{r}_T \times \underline{r}_{(T+90^\circ)}\right|} \qquad \text{(F-11)}$$

From the preceding equations, numerical values of the coordinate transformations (i. e. the orthogonal transformation matrix) between an ephemeris tabulation and the computational coordinates can be computed when the launch date and transfer time are specified.

APPENDIX G

COMPUTER PROGRAM

G. 1 Introduction

The FORTRAN program used to test the guidance theory is reproduced in this appendix. The input data format is given as well as the units for input data.

There are a few areas in the program which can be made more efficient, since testing is completed, however it operates quite satisfactorily as presently written. The author is indebted to Mr. Krupp for his magnificent efforts in writing this program.

G. 2 Input Format and Units

Nine data cards are required for each problem to be computed. The format for each is 3F20. 9. The following sequence and units are required:

1.) Initial position (A. u.)

2.) Initial velocity (A. u. /day)

3.) Target position (A. u.)

4.) Target velocity (A. u. /day)

5.) and 6.) Estimated $\underline{\nu}$ vector. For VSI control, both cards are null vectors. For CSI control, card number 5 is the null vector and card 6 should contain a number of the order of maximum initial acceleration. The units are (A. u. /day^2).

7.) a. Number of iterations desired
   b. Maximum initial acceleration in units of $10^{-4} g_o$. (Example, 1. 2). For VSI problems use any large number.
   c. Gravitational constant of the central body in units of $(A. u)^3/day^2$. For the sun this value is 0. 000295912.

8.) a. Maximum time step desired (days).

b. Earth gravity (A. u. /day^2). This value is 0.494840.

c. Flight time (days) .

9.) a. Exhaust power of engine per unit mass (A. u.)^2/day^3.
The conversion is: divide power in kw/kg by 3.4653×10^4.

b. Switching number. (use 1.0).

The routine will accept as many problems of one type as desired. CSI and VSI problems cannot be run together. Submit a complete set of data cards for each problem.

G.3 The FORTRAN Program

Except for subroutine DERV, the programs for CSI and VSI control are identical. There is a subroutine DERV for each mode of engine control. Select the routine appropriate to the problem and omit the other. Selection cannot be made automatically with the current program.

The FORTRAN source program is reproduced on the following pages. The generalized flow chart is presented in Figure G-a. Figure G-b is the storage map for computer variables.

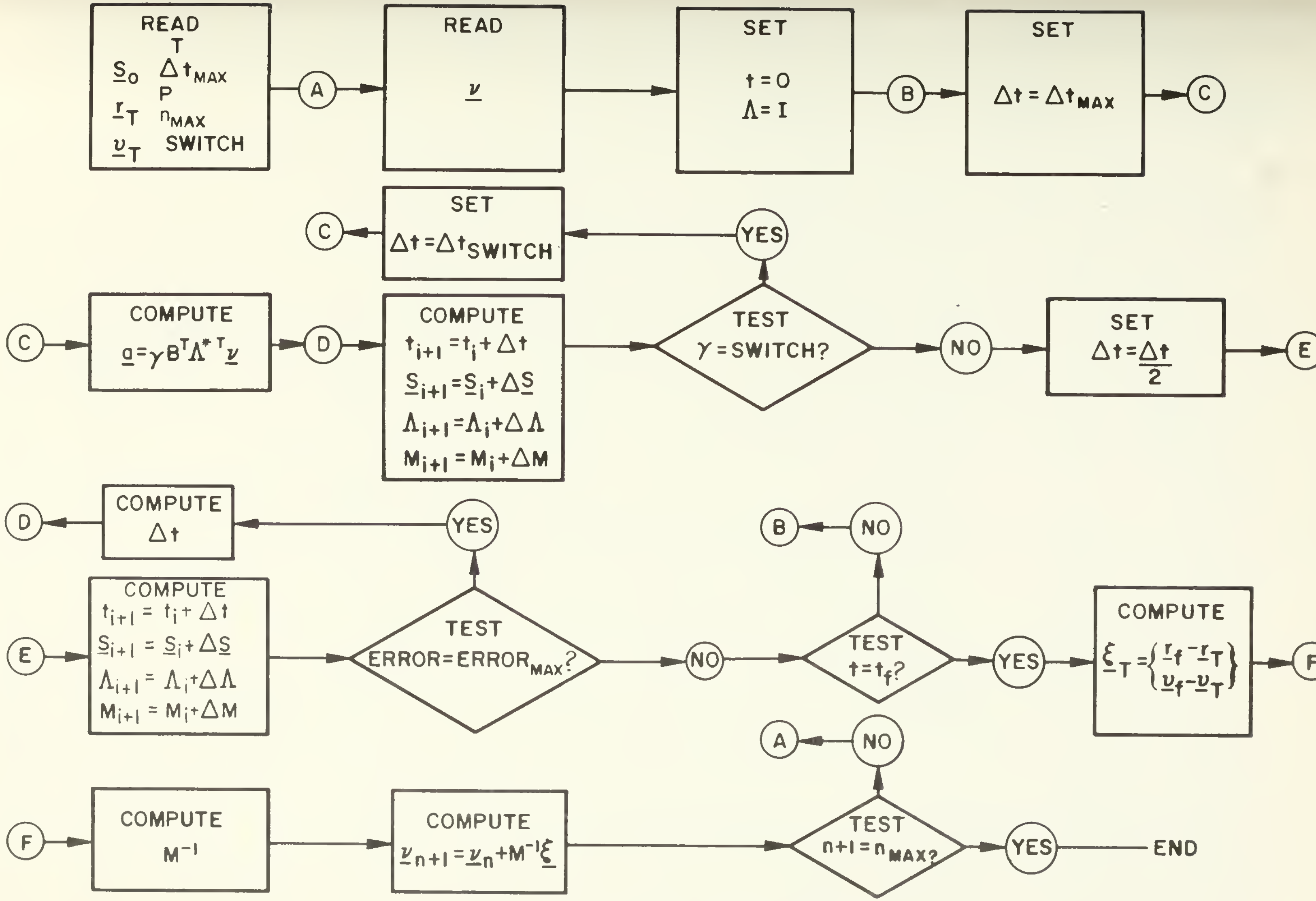

Fig. G-a. Flow chart to compute trajectories.

Fig. G-b. Computer storage map.

W_{ij}

	1	2	3	4	5	6	7	8	9	10	11	12	
j 1	r_1	λ_{r_1}	η_1		t	Λ_{11}	•	•	•	•	Λ_{61}	Λ_{71}	12
2	r_2	λ_{r_2}	η_2		$\int\lvert a\rvert$	•					•	•	24
3	r_3	λ_{r_3}	η_3		$\int a^2$	•					•	•	36
4	v_1	λ_{v_1}	η_4		$\int \frac{\gamma}{2} a^2$	•					•	•	48
5	v_2	λ_{v_2}	η_5		$\int m \lvert a\rvert$	•					•	•	60
6	v_3	λ_{v_3}	η_6		$\int \frac{\dot{a}}{m} \lvert a\rvert$	Λ_{16}	•	•	•	•	Λ_{66}	•	72
7	m	λ_m			Λ_m	Λ_{17}	•	•	•	•	•	Λ_{77}	84
8						M^*_{11}	•	•	•	•	M^*_{61}		96
9						•					•		108
10						•					•		120
11						•					•		132
12						•					•		144
13						M^*_{16}	•	•	•	•	M^*_{66}		156
14													168
15	r_{1E}	ξ_{1e}	r_{1T}	ξ_{1T}	ν_1	R	Flag	Switch					180
16	r_{2E}	ξ_{2e}	r_{2T}	ξ_{2T}	ν_2	γ	$\lvert\lambda_v\rvert m$	C					192
17	r_{3E}	ξ_{3e}	r_{3T}	ξ_{3T}	ν_3	a_0	μ	P					204
18	v_{1E}	ξ_{4e}	v_{1T}	ξ_{4T}	ν_4	a_1	Δ_{max}	D flag					216
19	v_{2E}	ξ_{5e}	v_{2T}	ξ_{5T}	ν_5	a_2	Error	Step Number					228
20	v_{3E}	ξ_{6e}	v_{3T}	ξ_{6T}	ν_6	a_3	Time	g_0					240

```
*     LABEL
*     LIST
C     MAIN
      DIMENSION W(12, 20), T(7, 8, 60)
    3 READ 1, ((W(I, J), J=15, 20), I=1, 7, 2), W(8, 17), W(8, 15)
C            INPUT DATA, (SEE FORMAT)
C            LAUNCH (POSITION, VELOCITY)
C            TARGET (POSITION, VELOCITY )
C            ESTIMATED NU VECTOR
C            NUMBER OF ITERATIONS, MAX ACCELERATION, GRAVITATION CONSTANT
C            MAX TIME STEP, EARTH GRAVITY, FLIGHT TIME
C            POWER, SWITCHING NUMBER
    1 FORMAT(3F20.9)
      NIT=W(7, 15)
      W(8, 20)=W(7, 19)/10000.0
      W(6, 17)=W(7, 16)*W(8, 20)
C            TRANSFER OF INPUT QUANTITIES TO FINAL STORAGE
      W(8, 16)=2.0*W(8, 17)/W(6, 17)
C            COMPUTATION OF EXHAUST VELOCITY
      DO 5 L=1. NIT
C            ITERATION LIMIT (NIT)
    5 CALL CORREC(W, T)
      CALL LAMTAB(W, T)
      PUNCH 2, (W(5, I), I=15, 20)
    2 FORMAT(3E20.8)
      GO TO 3
      END
                                                                        27
```

```
*      LAMTAB
*      LABEL
*      LIST
       SUBROUTINE LAMTAB(W, T)
C            SUBROUTINE COMPUTES, PRINTS BACKWARD INTEGRATED LAMBDA MATRIX
       DIMENSION W(12, 20), T(7, 8, 60), Q(10, 10)
       L=W(8, 19)+.01
C            L IS STEP NUMBER
       DO 20 LL=1, L
       A=T(7, 1, LL)*T(7, 1, L+1)
       IF(5*((LL-1)/5)-LL+1) 6, 5, 6
C            PAGE SPACING OF LAMBDA PRINT OUT
     5 PRINT 3
     6 DO 10 I=1, 7
       DO 10 J=1, 7
       Q(I, J)=0.0
       DO 10 K=1, 7
    10 Q(I, J)=Q(I, J)+T(I, K+1, LL)*T(K, J+1, L+1)
C            INVERTED FINAL VALUE OF FORWARD INTEGRATED LAMBDA MATRIX
C            MULTIPLIED INTO LAMBDA MATRIX AT EACH STEP
     4 FORMAT(104X, F16.8)
     3 FORMAT(1H12X4HTIME50X13HLAMBDA MATRIX)
     2 FORMAT(1H ,F7.2, 7F16.8)
     1 FORMAT(8X, 7F16.8)
       PRINT 2
       PRINT 2, T(1, 1, LL), (Q(I, 1), I=1, 7)
       PRINT 1, ((Q(I, J), I=1, 7), J=2, 7)
    20 PRINT 4, A
       RETURN
       END
                                                                        30
```

```
*       CORREC
*       LABEL
*       LIST
        SUBROUTINE CORREC(W, T)
C             SUBROUTINE COMPUTES NEW INITIAL VALUES FOR NEXT ITERATION
C             AND PRINTS FINAL VALUES OF INTEGRATED QUANTITIES
        DIMENSION W(12, 20), Q(10, 10), T(7, 8, 50)
        CALL INTEG(W, T)
        DO 10 I=1, 6
        W(8, 15)=2.0
        W(4, I+14)=W(1, I)-W(3, I+14)
C             COMPUTATION OF MISS VECTOR AT TARGET (TARGET XI)
        W(2, I+14)=0.0
        DO 10 J=1, 6
        W(2, I+14)=W(2, I+14)+W(I+5, J)*(W(1, J)-W(3, J+14))
C             TRANSFORMS TARGET XI TO EQUIVALENT LAUNCH ERROR (INITIAL XI)
     10 Q(I, J)=W(I+5, J+7)
        CALL INVERT(6, Q)
C             INVERSION OF M STAR MATRIX
        DO 20 I=1, 6
        DO 20 J=1, 6
     20 W(5, I+14)=W(5, I+14)-Q(I, J)*W(2, J+14)
C             COMPUTATION OF CORRECTED NU VECTOR
      3 FORMAT(1H15X13HINITIAL STATE5X11HFINAL STATE6X12HTARGET STATE9X
     C 3HETA11X9HTARGET XI8X10HINITIAL XI7X10HNU X 10000)
      2 FORMAT(1H  7E17.9)
      1 FORMAT(1H0/1H09X1HJ53X12HMSTAR MATRIX/1H0E17.7,6F17.6/(16X6F17.6))
        AJ=W(5, 3)/2.0
        PRINT 3
        PRINT 2
        DO 25 I=15, 20
        XNU=W(5, I)*10000.0
     25 PRINT 2, W(1,I),W(1,I-14),W(3,I),W(3,I-14),W(4,I),W(2,I),XNU
        PRINT 1, AJ, ((W(I, J),I=6, 11), J=8, 13)
        DO 30 I=1, 7
        DO 30 J=1, 7
     30 Q(I, J)=W(J+5, I)
        CALL INVERT(7, Q)
        L=W(8, 19)+1.01
        DO 40 I=1, 7
        DO 40 J=2, 8
     40 T(I, J, L)=Q(I, J-1)
        T(7, 1, L)=1.0/W(5, 7)
C             INVERSION OF FINAL VALUE OF FORWARD INTEGRATED LAMBDA MATRIX
        RETURN
        END
```

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```
*     INTEG
*     LABEL
*     LIST
      SUBROUTINE INTEG(W, T)
C           SUBROUTINE CONTROLS INITIAL VALUES FOR INTEGRATION
C           AND PRINTS OUT TRAJECTORY VARIABLES AT EACH STEP
      DIMENSION W(12, 20), D(12, 20), T(7, 8, 50)
      DO 10 I=1, 168
      D(I, 1)=0.0
   10 W(I, 1)=0.0
      DO 20 I=1, 6
      W(I+5, I)=1.0
      W(1, I)=W(1, I+14)
   20 W(2, I)=W(5, I+14)
      W(1, 7)=1.0
      W(2, 7)=0.0
      W(5, 7)=1.0
      W(12, 7)=1.0
      W(7, 19)=0.0
      W(7, 15)=1.0
      W(8, 18)=0.0
C           INITIAL CONDITIONS FOR INTEGRATION
      CALL DERV(W, D)
      IF(W(7, 16)-W(6, 17)) 30, 30, 35
C           TEST FOR THRUST LIMITATION SWITCH POINT
   30 W(7, 15)=0.0
      CALL DERV(W, D)
   35 DEL=W(7, 18)
C           LIMITATION OF INTEGRATION STEP SIZE
    3 FORMAT(1H111X4HTIME8X12HACCELERATION7X9HACCEL MAG8X8HPOSITION9X
     C 8HVELOCITY11X4HMASS13X5HGAMMA/26X9HX 10000/G8X9HX 10000/G)
    2 FORMAT(1H .F17.4, 6F17.7)
    1 FORMAT(1H ,17X, F17.7, 17X, 2F17.7)
      DO 70 I=1, 50
      A1=W(6, 18)/W(8, 20)
      A2=W(6, 19)/W(8, 20)
      A3=W(6, 20)/W(8, 20)
      AM=D(5, 2)/W(8, 20)
      GAM=W(7, 16)/W(6, 17)
      W(8, 19)=I
      DO 40 J=1, 8
      DO 40 K=1, 7
   40 T(K, J, I)=W(J+4, K)
      IF(10*((I-1)/10)-I+1) 46, 45, 46
C           PAGE SPACING FOR TRAJECTORY DATA
   45 PRINT 3
   46 PRINT 2
      PRINT 2, W(5,1),A1,AM,W(1,1),W(1,4),W(1,7),GAM
      PRINT 1, A2,W(1,2),W(1,5),A3,W(1,3),W(1,6)
      IF(DEL+W(5, 1)-W(7, 20)) 60, 60, 50
C           TEST FOR END OF FLIGHT TIME
   50 DEL=W(7, 20)-W(5, 1)
   60 IF(DEL/W(7, 18)-.0001) 80, 80, 70
C           TEST FOR FINAL STEP LESS THAN .0001
   70 CALL STEP(W, D, DEL)
   80 RETURN
      END
```

57

```
*      STEP
*      LABEL
*      LIST
       SUBROUTINE STEP(W1, D1, DEL)
C            SUBROUTINE SELECTS STEP SIZE ON BASIS OF INTEGRATION
C            ERROR AND THRUST LIMITATION SWITCH POINT
       DIMENSION W1(240), D1(240), W2(240), D2(240), W3(240), D3(240)
     5 FLAG=W1(175)
C            DEFINITION OF FLAG
       DO 10 I=1, 240
       D2(I)=D1(I)
       D3(I)=D1(I)
       W2(I)=W1(I)
    10 W3(I)=W1(I)
C            DEFINITION OF VARIABLES FOR TRIAL INTEGRATION
       CALL RUNKUT(W2, D2, DEL)
C            TRIAL INTEGRATION
       IF(FLAG-W2(186)) 20, 15, 15
C            TEST FOR THRUST LIMIT SWITCH OFF POINT DURING INTEGRATION
    15 FLAG=0.0
       GO TO 30
    20 IF(FLAG-MIN1F(W2(186)-1.0, .5)) 25, 40, 40
C            TEST FOR THRUST LIMIT SWITCH ON POINT DURING INTEGRATION
    25 FLAG=1.0
    30 DEL=DEL*ABSF((W1(187)-W1(198))/(W1(187)-W2(187)))
C            SELECTION OF STEP BEGINNING AT SWITCH POINT
       DO 35 I=1, 240
       D2(I)=D1(I)
    35 W2(I)=W1(I)
       CALL RUNKUT(W2, D2, DEL)
C            INTEGRATION OVER FULL STEP
    40 CALL RUNKUT(W3, D3, DEL/2.0)
       CALL RUNKUT(W3, D3, DEL/2.0)
C            TWO INTEGRATIONS OF HALF STEP EACH
    45 TEST=ABSF(W2(174)/W3(174)-1.0)/.000045+.0001
       IF(TEST-2.0) 55, 55, 50
C            DIFFERENCE TEST OF FULL STEP AND HALF STEP INTEGRATION
    50 DEL=DEL/TEST**.25
    55 IF(TEST-.05) 60, 60, 70
    60 DEL=MIN1F(W1(211), DEL/TEST**.25)
C            SELECTION OF STEP SIZE FROM ERROR TEST
    70 DO 75 I=1, 240
       D1(I)=D3(I)
    75 W1(I)=W3(I)
C            STORAGE OF INTEGRATED VALUES FOR NEXT INTEGRATION STEP
       W1(223)=TEST*.000003
       IF(W1(175)-FLAG) 80, 90, 80
    80 W1(212)=W1(175)-FLAG
       W1(175)=FLAG
C            STORAGE OF THRUST LIMIT SWITCH POINT
       CALL DERV(W1, D1)
       W1(212)=0.0
    90 RETURN
       END
```

54

```
*      RUNKUT
*      LABEL
*      LIST
       SUBROUTINE RUNKUT(W1, D1, DEL)
C            SUBROUTINE PERFORMS INTEGRATION USING KUTTAS SIMPSONS RULE
C            LET X* = DX/DT
C                X* = F(X,T)
C             (XO)* = F(XO, TO)
C             (X1)* = F( (XO + .5(DEL T)(XO)*), (TO + .5(DEL T)) )
C             (X2)* = F( (XO + .5(DEL T)(X1)*), (TO + .5(DEL T)) )
C             (X3)* = F(XO + (DEL T)(X2)*), (TO + DEL T) )
C             DEL X = (DEL T)( (XO)* + 2(X1)* + 2(X2)* + (X3)*)/6
       DIMENSION W1(240), D1(168), W(240), D(168, 4), C(3)
       C(1)=.5
       C(2)=.5
       C(3)=1.0
       DO 5 I=1, 168
       W(I+72)=W1(I+72)
     5 D(I, 1)=D1(I)
       DO 15 L=1, 3
       E=DEL*C(L)
       DO 10 I=1, 168
    10 W(I)=W1(I)+E*D(I, L)
    15 CALL DERV(W, D(1, L+1))
       E=DEL/6.0
       DO 20 I=1, 168
    20 W1(I)=W1(I)+E*(D(I, 1)+2.0*(D(I, 2)+D(I, 3))+D(I, 4))
       CALL DERV(W1, D1)
       RETURN
       END
```

30

```
*       INVERT
*       LABEL
*       LIST
        SUBROUTINE INVERT(N, QQ)
C             SUBROUTINE INVERTS MATRIX BY SIMULTANEOUS DOUBLE PRECISION
C             ROW REDUCTION OF THE MATRIX TO IDEM AND IDEM TO THE INVERSE
D       DIMENSION QQ(10, 10), Q(10, 20)
        DO 10 I=1, 10
        DO 5 J=1, 10
        Q(I, J)=QQ(I, J)
        Q(I, J+10)=0.0
        Q(I, J+20)=0.0
      5 Q(I, J+30)=0.0
     10 Q(I, I+10)=1.0
        DO 30 I=1, N
        DO 14 J=I, N
        IF(ABSF(Q(I, I))-ABSF(Q(J, I))) 11, 14, 14
C             TEST FOR LARGEST ELEMENT IN COLUMN
     11 DO 12 K=1, N
D       S=Q(J, K)
D       Q(J, K)=Q(I, K)
D       Q(I, K)=S
D       S=Q(J, K+10)
D       Q(J, K+10)=Q(I, K+10)
D    12 Q(I, K+10)=S
C             TRANSFER ROW OF LARGEST ELEMENT TO FIRST ROW
     14 CONTINUE
D       DIV=Q(I, I)
        DO 15 J=1, N
D       Q(I, J)=Q(I, J)/DIV
D    15 Q(I, J+10)=Q(I, J+10)/DIV
C             DIVISION BY DIAGONAL ELEMENTS
        DO 30 J=1, N
        IF(I-J) 20, 30, 20
D    20 DIV=Q(J, I)
        DO 25 K=1, N
D       Q(J, K)=Q(J, K)-Q(I, K)*DIV
D    25 Q(J, K+10)=Q(J, K+10)-Q(I, K+10)*DIV
C             DIAGONALIZATION OF MATRIX
     30 CONTINUE
        DO 35 I=1, N
        DO 35 J=1, N
     35 QQ(I, J)=Q(I, J+10)
        RETURN
        END
```

45

TOTAL 289*

```
*       DERV
*       LABEL
*       LIST
        SUBROUTINE DERV(W, D1)
C       LINEAR OPTIMUM
C             SUBROUTINE ESTABLISHES DIFFERENTIAL EQUATIONS OF SYSTEM
        DIMENSION W(12, 20), D(12, 14), D1(168)
        R=SQRTF(W(1, 1)**2+W(1, 2)**2+W(1, 3)**2)
        RM3=W(7, 17)/R**3
        RM5=3.0*RM3/R**2
        RL=RM5*(W(1, 1)*W(2, 4)+W(1, 2)*W(2, 5)+W(1, 3)*W(2, 6))
C             COMPUTATION OF GRAVITATIONAL GRADIENT
        A2=SQRTF(W(2, 4)**2+W(2, 5)**2+W(2, 6)**2)
        D(5, 1)=1.0
        D(5, 2)=W(6, 17)/W(1, 7)*W(7, 15)
        D(5, 3)=D(5, 2)**2
        D(5, 4)=.5*A2*D(5, 2)
        D(5, 5)=D(5, 2)*W(1, 7)
        D(5, 6)=D(5, 2)/W(1, 7)*W(6, 17)
C             DERIVATIVES OF PARAMETRIC VARIABLES
        W(6, 15)=R
C             STORAGE OF R FOR ERROR TEST
        W(6, 16)=MAX1F(A2*W(1, 7)/W(6, 17)*W(8, 15), 1.0)
        W(7, 16)=A2*W(1, 7)*W(8, 15)
C             COMPUTATION OF THRUST SWITCHING FUNCTION
        DO 10 I=1, 3
        W(6, I+17)=W(2, I+3)*D(5, 2)/A2
C             COMPUTATION OF ACCELERATION VECTOR
        D(1, I)=W(1, I+3)
        D(1, I+3)=W(6, I+17)-RM3*W(1, I)
C             EQUATIONS OF MOTION
        D(2, I)=RM3*W(2, I+3)-RL*W(1, I)
   10   D(2, I+3)=-W(2, I)
C             EULER EQUATIONS (MASS INDEPENDENT)
        D(1, 7)=-W(6, 17)/W(8, 16)*W(7, 15)
C             MASS RATE EQUATION
        D(2, 7)=W(2, 7)*D(5, 2)/W(8, 16)
C             EULER EQUATION FOR MASS
        D(5, 7)=W(5, 7)*D(5, 2)/W(8, 16)
C             ADJOINT EQUATION FOR MASS SENSITIVITY
        DO 20 I=1, 6
        D(3, I)=W(I+5, 4)*W(6, 18)+W(I+5, 5)*W(6, 19)+W(I+5, 6)*W(6, 20)
C             EQUATION FOR ETA VECTOR
        W(4, I)=W(I+5, 4)*W(2, 4)+W(I+5, 5)*W(2, 5)+W(I+5, 6)*W(2, 6)
C             QUANTITY USED IN M STAR MATRIX
        RL=RM5*(W(I+5, 4)*W(1, 1)+W(I+5, 5)*W(1, 2)+W(I+5, 6)*W(1, 3))
        D(I+5, 7)=D(3, I)/W(1, 7)
C             ADJOINT EQUATIONS FOR MASS DEPENDENCE
        DO 20 J=1, 3
        D(I+5, J)=RM3*W(I+5, J+3)-RL*W(1, J)
   20   D(I+5, J+3)=-W(I+5, J)
C             ADJOINT EQUATIONS (MASS INDEPENDENT)
```

52

TOTAL 52*

```
      RL=(W(2, 1)*W(2, 4)+W(2, 2)*W(2, 5)+W(2, 3)*W(2, 6))*A2*W(1, 7)
C           QUANTITY USED IN M STAR MATRIX
      DO 30 I=1, 6
      DO 30 J=I, 6
      F=0.0
      DO 25 K=1, 3
   25 F=F+W(I+5, K+3)*W(J+5, K+3)
      D(I+5, J+7)=(F-D(3, I)*D(3, J)/D(5, 3))*D(5, 2)/A2
C           DERIVATIVE OF M STAR MATRIX
      W(I+5, J+7)=W(I+5, J+7)+W(4, I)*W(4, J)/RL*W(6, 17)*W(8, 18)
C           CORRECTION TO M STAR FOR VARIABLE INTEGRATION TIME
      W(J+5, I+7)=W(I+5, J+7)
   30 D(J+5, I+7)=D(I+5, J+7)
C           M STAR IS SYMMETRIC
      DO 40 I=1, 168
   40 D1(I)=D(I, 1)
      RETURN
      END
                                                                  18

                                                     TOTAL        18*
```

```
*       DERV
*       LABEL
*       LIST
        SUBROUTINE DERV(W, D1)
C       QUADRATIC OPTIMUM
C            SUBROUTINE ESTABLISHES DIFFERENTIAL EQUATIONS OF SYSTEM
        DIMENSION W(12, 20), D(12, 14), D1(168)
        W(8, 15)=1.0
        R=SQRTF(W(1, 1)**2+W(1, 2)**2+W(1, 3)**2)
        RM3=W(7, 17)/R**3
        RM5=3.0*RM3/R**2
        RL=RM5*(W(1, 1)*W(2, 4)+W(1, 2)*W(2, 5)+W(1, 3)*W(2, 6))
C            COMPUTATION OF GRAVITATIONAL GRADIENT
        A2=SQRTF(W(2, 4)**2+W(2, 5)**2+W(2, 6)**2)
        GMASS=W(2, 7)*W(1, 7)**2/W(8, 17)
        GAM=MAX1F(A2/W(6, 17)*W(1, 7)-GMASS, 1.0)
C            THRUST LIMIT SWITCHING FUNCTION
        STAR=W(7, 15)*(W(1, 7)/W(6, 17))**2
C            QUANTITY USED IN M STAR MATRIX
        D(5, 1)=1.0
        D(5, 2)=A2/(GAM+GMASS)
        D(5, 3)=D(5, 2)**2
        D(5, 4)=.5*GAM*D(5, 3)
        D(5, 5)=D(5, 2)*W(1, 7)
        D(5, 6)=D(5, 2)/W(1, 7)*W(6, 17)
C            DERIVATIVES OF PARAMETRIC VARIABLES
        W(6, 15)=R
C            STORAGE OF R FOR ERROR TEST
        W(6, 16)=GAM
        W(7, 16)=A2*W(1, 7)-GMASS*W(6, 17)
C            STORAGE OF THRUST LIMIT FUNCTION
        DO 10 I=1, 3
        W(6, I+17)=W(2, I+3)/(GAM+GMASS)
C            COMPUTATION OF ACCELERATION VECTOR
        D(1, I)=W(1, I+3)
        D(1, I+3)=W(6, I+17)-RM3*W(1, I)
C            EQUATIONS OF MOTION
        D(2, I)=RM3*W(2, I+3)-RL*W(1, I)
   10   D(2, I+3)=-W(2, I)
C            EULER EQUATIONS (MASS INDEPENDENT)
        D(1, 7)=-D(5, 3)*W(1, 7)**2/W(8, 17)/2.0
C            MASS RATE EQUATION
        D(2, 7)=W(1, 7)*D(5, 3)/W(8, 17)*W(2, 7)
C            EULER EQUATION FOR MASS
        D(5, 7)=W(1, 7)*D(5, 3)/W(8, 17)*W(5, 7)
C            ADJOINT EQUATION FOR MASS SENSITIVITY
        DO 20 I=1, 6
        D(3, I)=W(I+5, 4)*W(6, 18)+W(I+5, 5)*W(6, 19)+W(I+5, 6)*W(6, 20)
C            EQUATION FOR ETA VECTOR
        RL=RM5*(W(I+5, 4)*W(1, 1)+W(I+5, 5)*W(1, 2)+W(I+5, 6)*W(1, 3))
        D(I+5, 7)=D(3, I)/W(1, 7)
C            ADJOINT EQUATIONS OF MASS DEPENDENCE
                                                                          52

                                                          TOTAL          52*
```

```
      DO 20 J=1, 3
      D(I+5, J)=RM3*W(I+5, J+3)-RL*W(1, J)
   20 D(I+5, J+3)=-W(I+5, J)
C           ADJOINT EQUATIONS (MASS INDEPENDENT)
      DO 30 I=1, 6
      DO 30 J=I, 6
      F=0.0
      DO 25 K=1, 3
   25 F=F+W(I+5, K+3)*W(J+5, K+3)
      D(I+5, J+7)=(F-D(3, I)*D(3, J)*STAR)/GAM
C           DERIVATIVE OF M STAR MATRIX
   30 D(J+5, I+7)=D(I+5, J+7)
C           M STAR IS SYMMETRIC
      DO 40 I=1, 168
   40 D1(I)=D(I, 1)
      RETURN
      END
                                                                  17

                                                       TOTAL      17*
```

APPENDIX H

NUMERICAL RESULTS

H. 1 Introduction

In this appendix the computer output data from a sample run of each control mode are reproduced. The data sheets are in general self explanatory except perhaps for the format of the Λ matrix and the units.

All parameters and variables derived in the linear theory are printed out. In addition, the important variables are plotted in Figures H-a through H-r.

H. 2 Format and Units of the Data

The first pages of each example contain:

1.) The time in days from initial point,

2.) The acceleration program in units of $10^{-4} g_o$,

3.) The acceleration magnitude in units of $10^{-4} g_o$,

4.) The position in A. u.,

5.) The velocity in A. u. per day,

6.) The normalized mass,

7.) The switching parameter, γ.

The coordinates are computational coordinates in all cases.

Following the state and acceleration data are the terminal values of interest printed as column vectors. The units are A. u. and A. u. per day.

The elements of the M* matrix, which are the next set of numbers, are not particularly interesting and may be disregarded.

Adjacent to the M* matrix is the cost quantity J with units of $(\text{A. u.})^2$ per $(\text{day})^3$.

The next set of pages contain the elements of the Λ matrix for both optimal acceleration and optimal thrust programs. For each time point, the array consists of 50 elements which are interpreted as partial derivatives or influence coefficients. The array is ordered as in the equation

$$\delta \underline{s}_f = \Lambda(t)\, \delta \underline{s}_t$$

The first 49 elements are the adjoint set for an optimal thrust program. The adjoint set for an optimal acceleration program is obtained by replacing the seventh column with zeros for the first six elements, and replacing the 1 with the element below it. That is: the array

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{\Lambda}_{13} \\ \Lambda_{21} & \Lambda_{22} & \underline{\Lambda}_{23} \\ \underline{O}^T & \underline{O}^T & 1 \\ & & \Lambda_{33} \end{bmatrix}$$

yields $\underline{\Lambda \text{ (optimal thrust)}}$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{\Lambda}_{13} \\ \Lambda_{21} & \Lambda_{22} & \underline{\Lambda}_{23} \\ \underline{O}^T & \underline{O}^T & 1 \end{bmatrix}$$

and $\underline{\Lambda \text{ (optimal acceleration)}}$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \underline{O} \\ \Lambda_{21} & \Lambda_{22} & \underline{O} \\ \underline{O}^T & \underline{O}^T & \Lambda_{33} \end{bmatrix}$$

The applicable submatrices, interpreted as partial derivatives, are

$$\begin{bmatrix} \dfrac{\partial \underline{r}_f}{\partial \underline{r}_t} & \dfrac{\partial \underline{r}_f}{\partial \underline{v}_t} & \dfrac{\partial \underline{r}_f}{\partial m_t} \\ \dfrac{\partial \underline{v}_f}{\partial \underline{r}_t} & \dfrac{\partial \underline{v}_f}{\partial \underline{v}_t} & \dfrac{\partial \underline{v}_f}{\partial m_t} \\ \dfrac{\partial m_f}{\partial \underline{r}_t} & \dfrac{\partial m_f}{\partial \underline{v}_t} & \dfrac{\partial m_f}{\partial m_t} \end{bmatrix}$$

The units are A. u. and A. u. per day for position and velocity. The seventh column is with respect to a 100% change in mass. For example the units of $\dfrac{\partial \underline{r}_f}{\partial m_t}$ are $\dfrac{\text{A. u.}}{\text{100\% change in mass}}$.

The first set of data is for unrestricted VSI control. The second set is for CSI control with $a_o = 1.2 \times 10^{-4} g_o$.

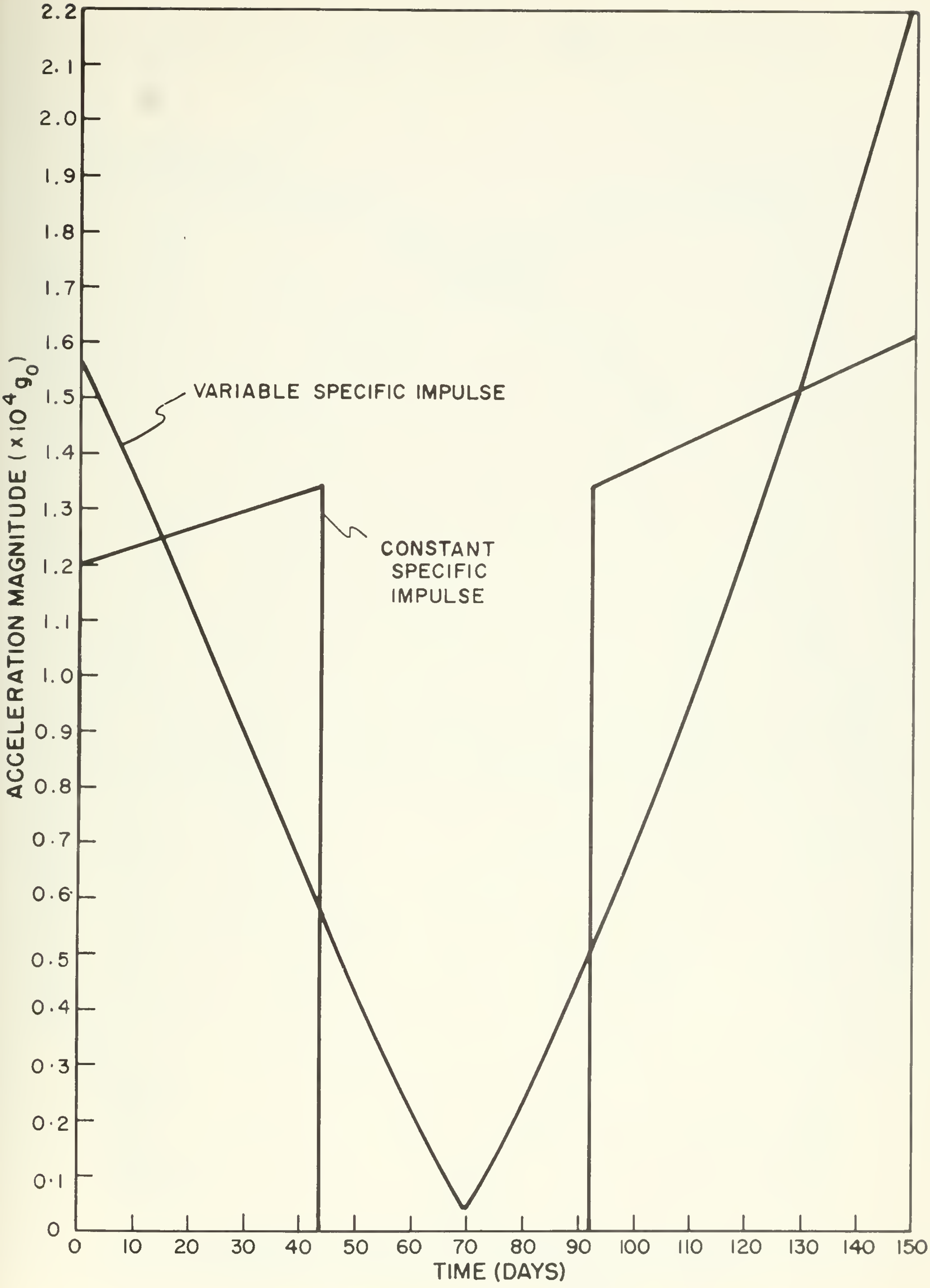

Fig. H-a. Acceleration schedule for 150-day Earth-Mars transfer.

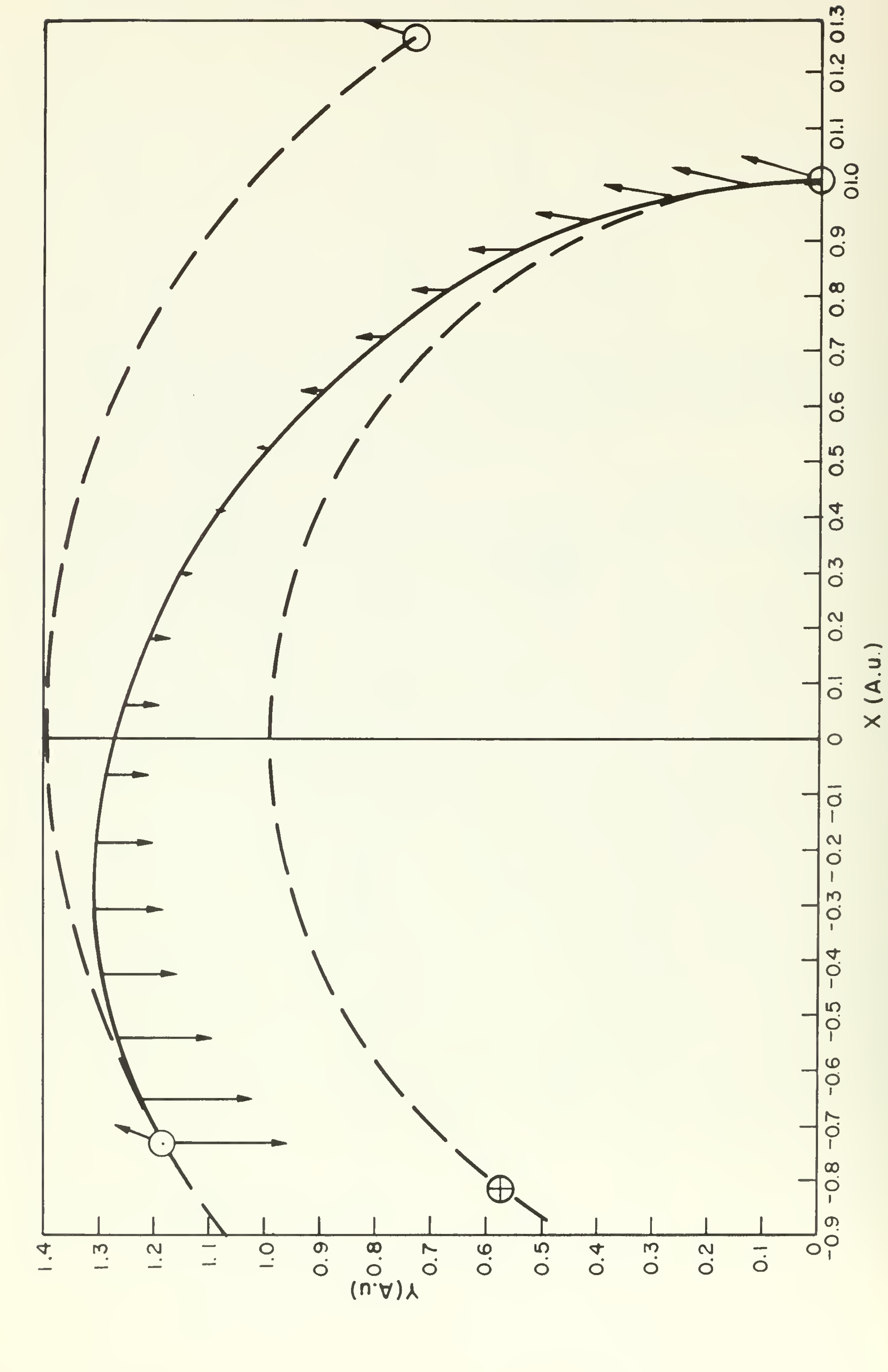

Fig. H-b. Transfer plane projection of variable-specific-impulse optimal transfer.

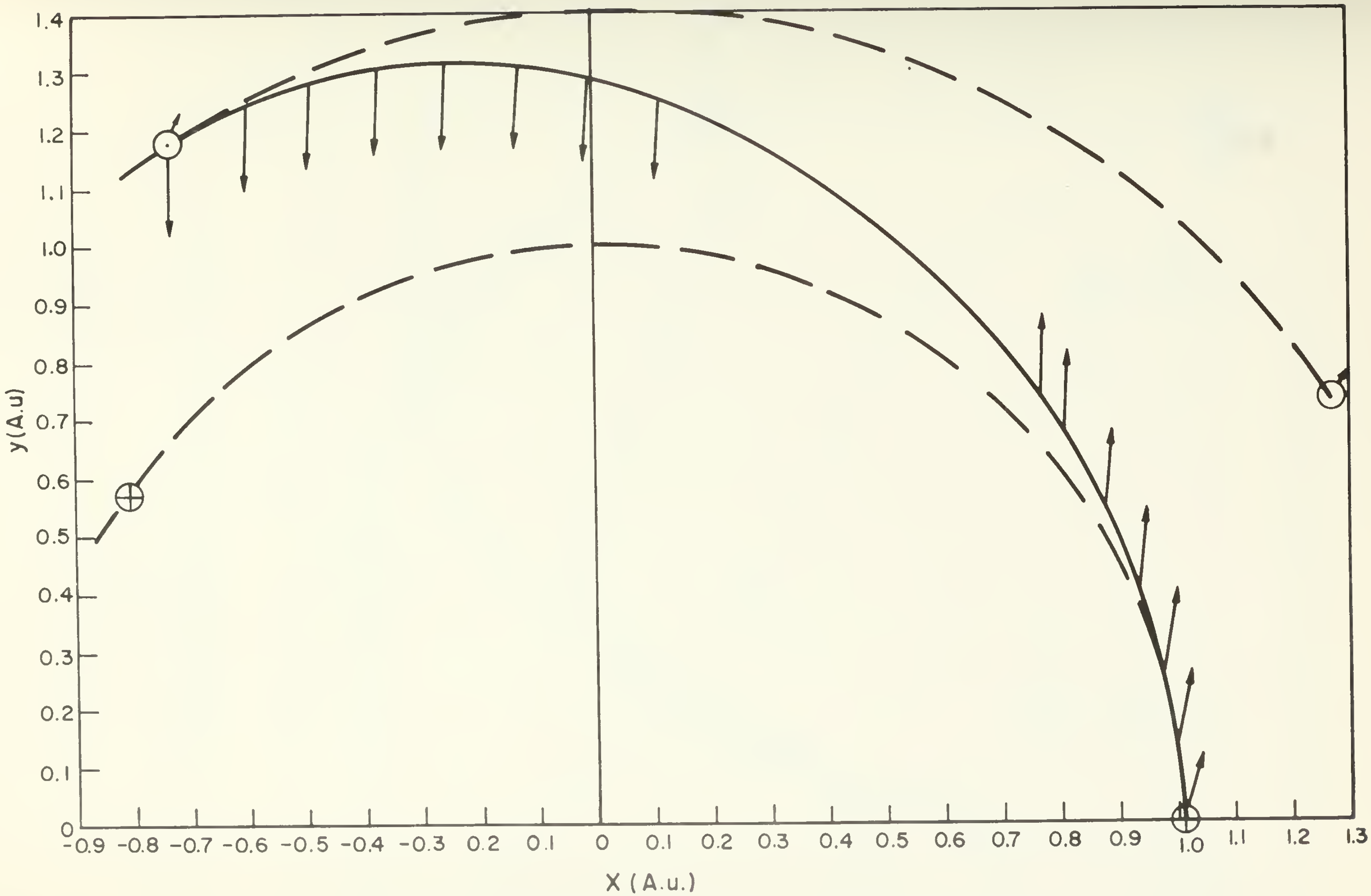

Fig. H-c. Transfer plane projection of constant-specific-impulse optimal trajectory.

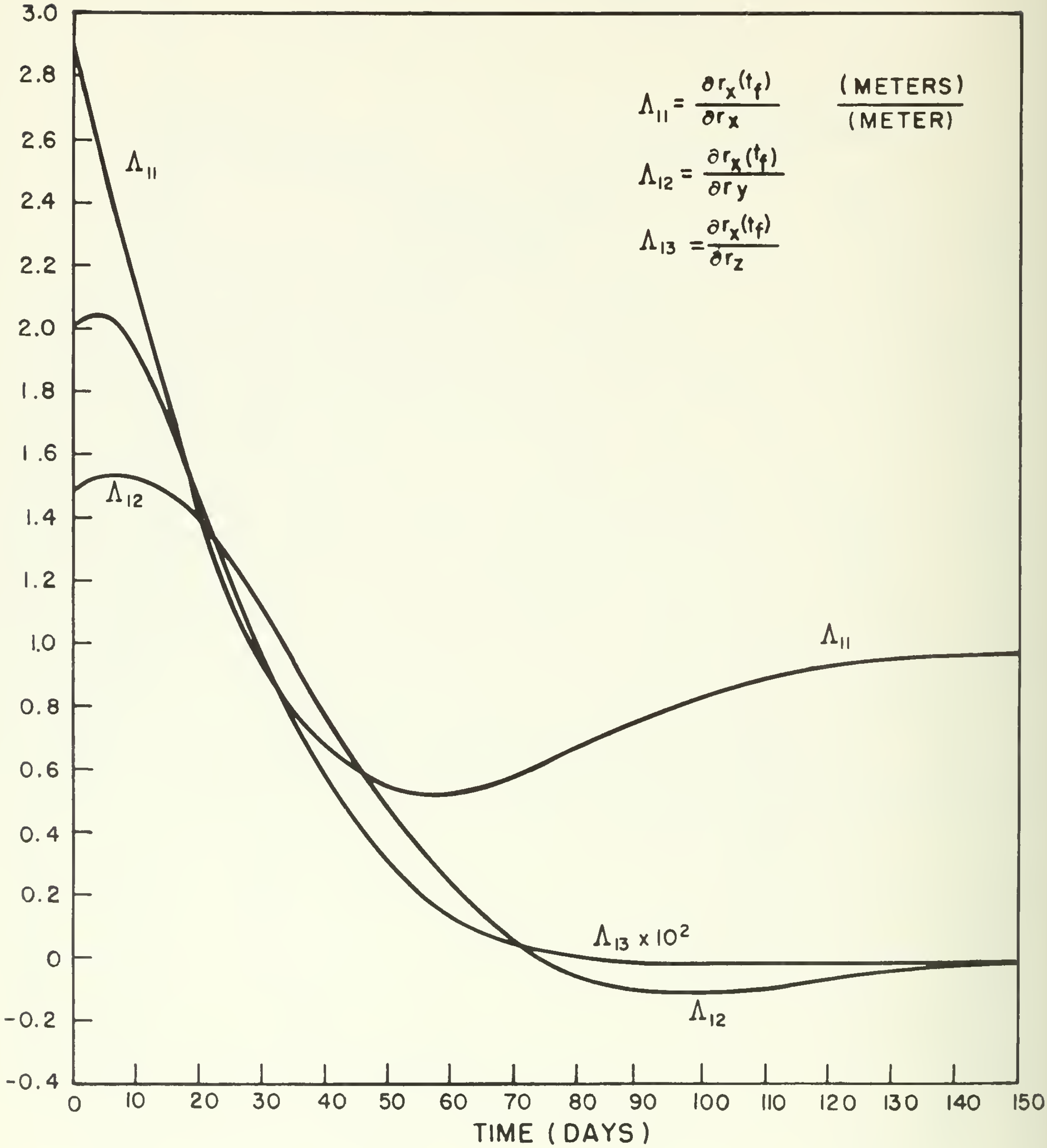

Fig. H-d. Sensitivity of r_x (t_f) to position variations.

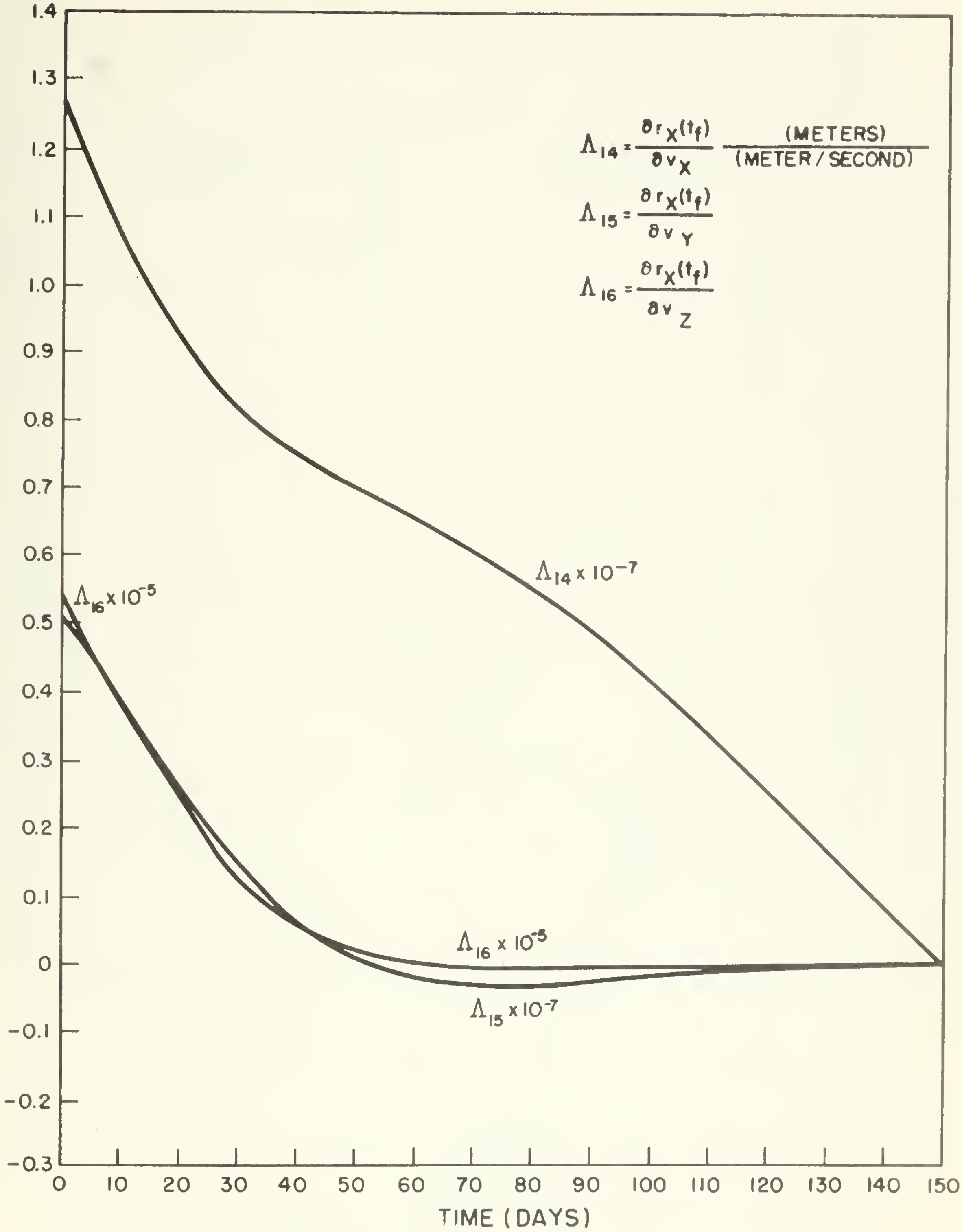

Fig. H-e. Sensitivity of r_X (t_f) to velocity variations.

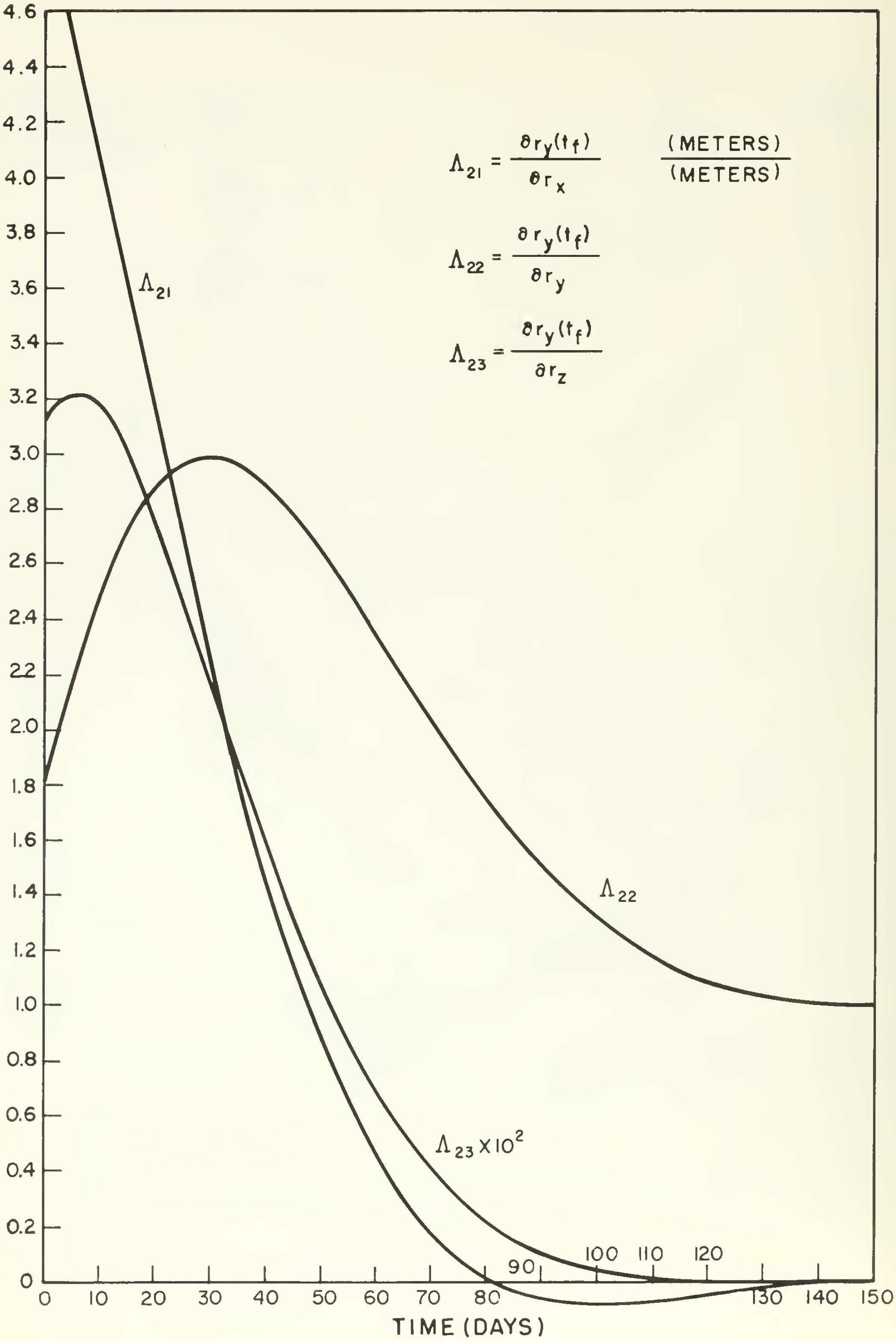

Fig. H-f. Sensitivity of r_y (t_f) to position variations.

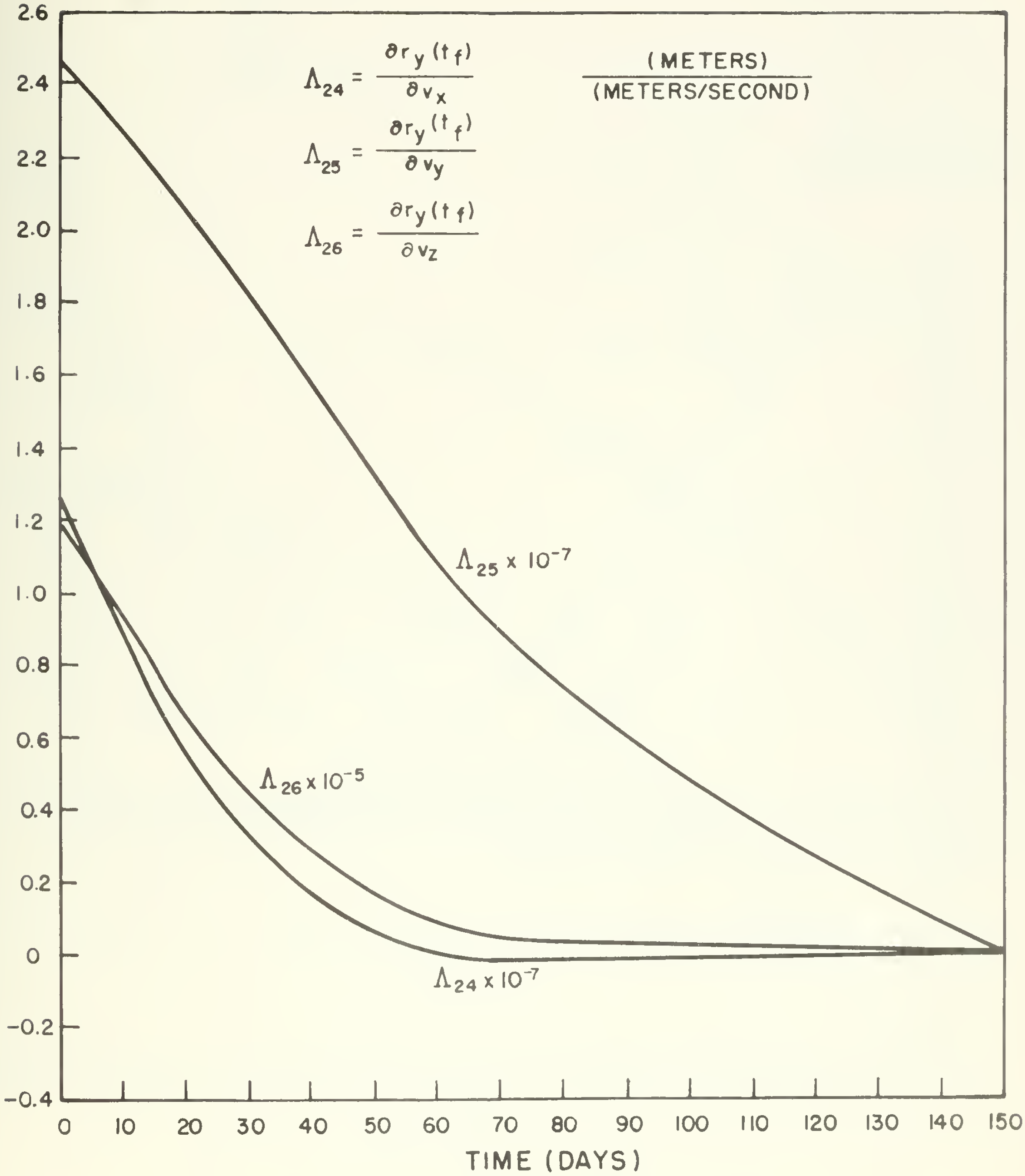

Fig. H-g. Sensitivity of r_y (t_f) to velocity variations.

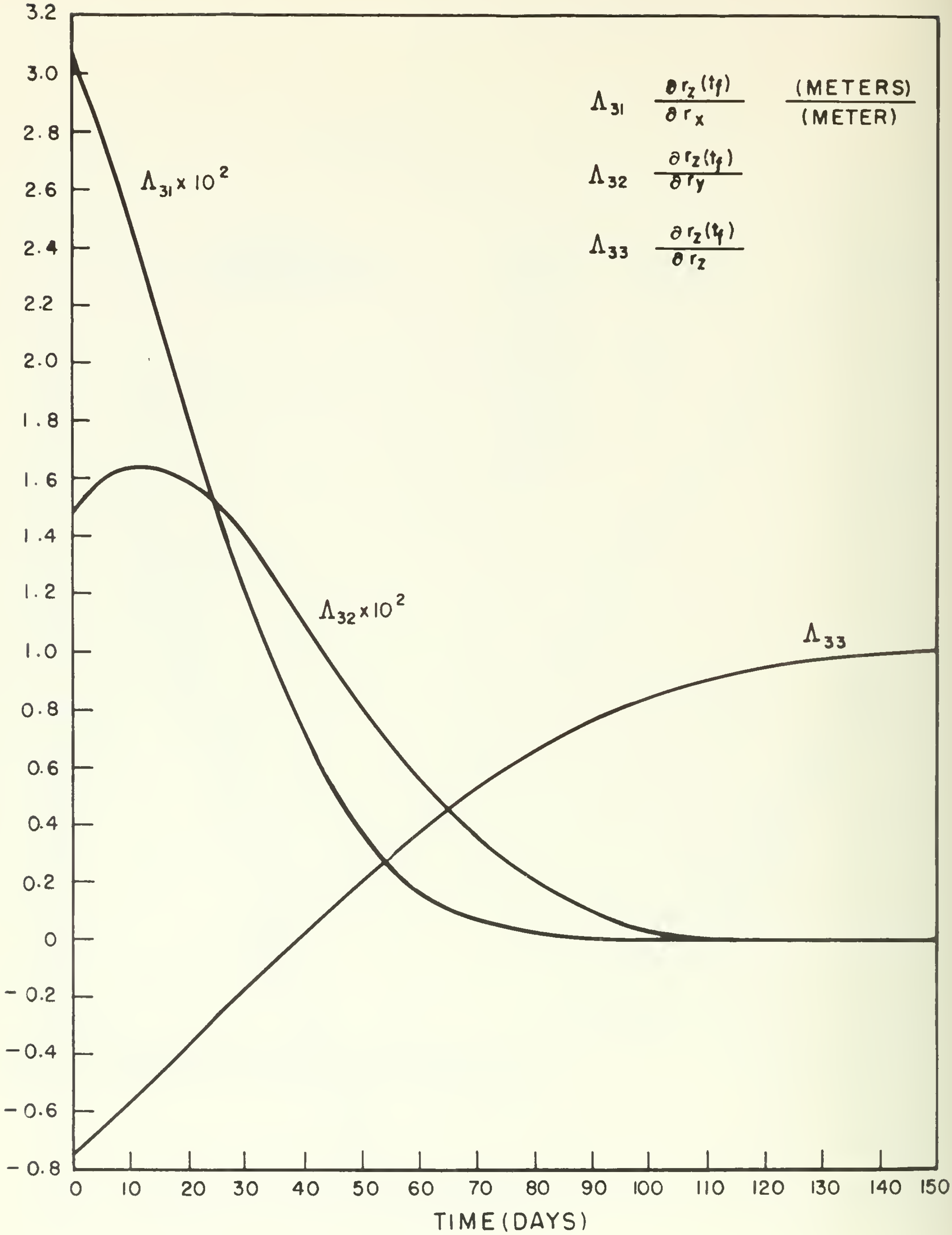

Fig. H-h. Sensitivity of r_z (t_f) to position variations.

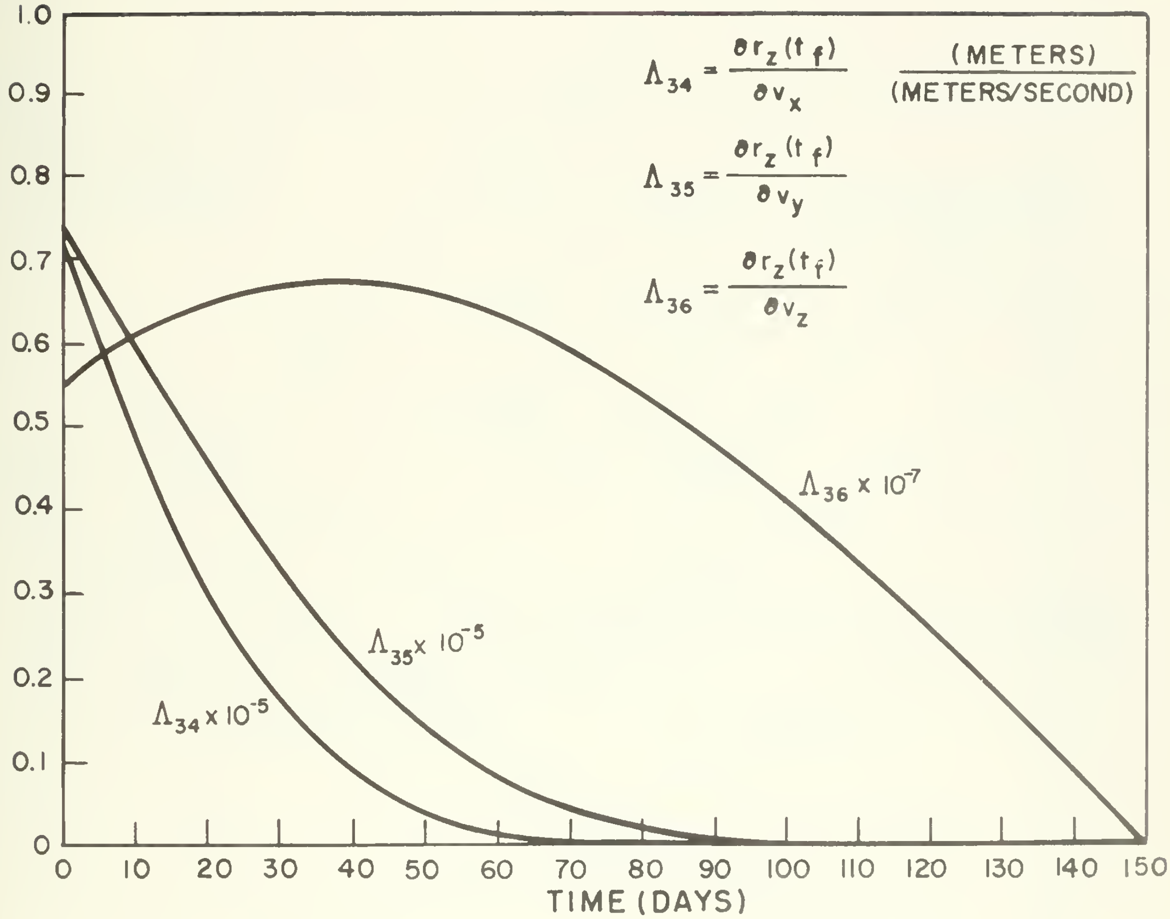

Fig. H-i. Sensitivity of r_z (t_f) to velocity variations.

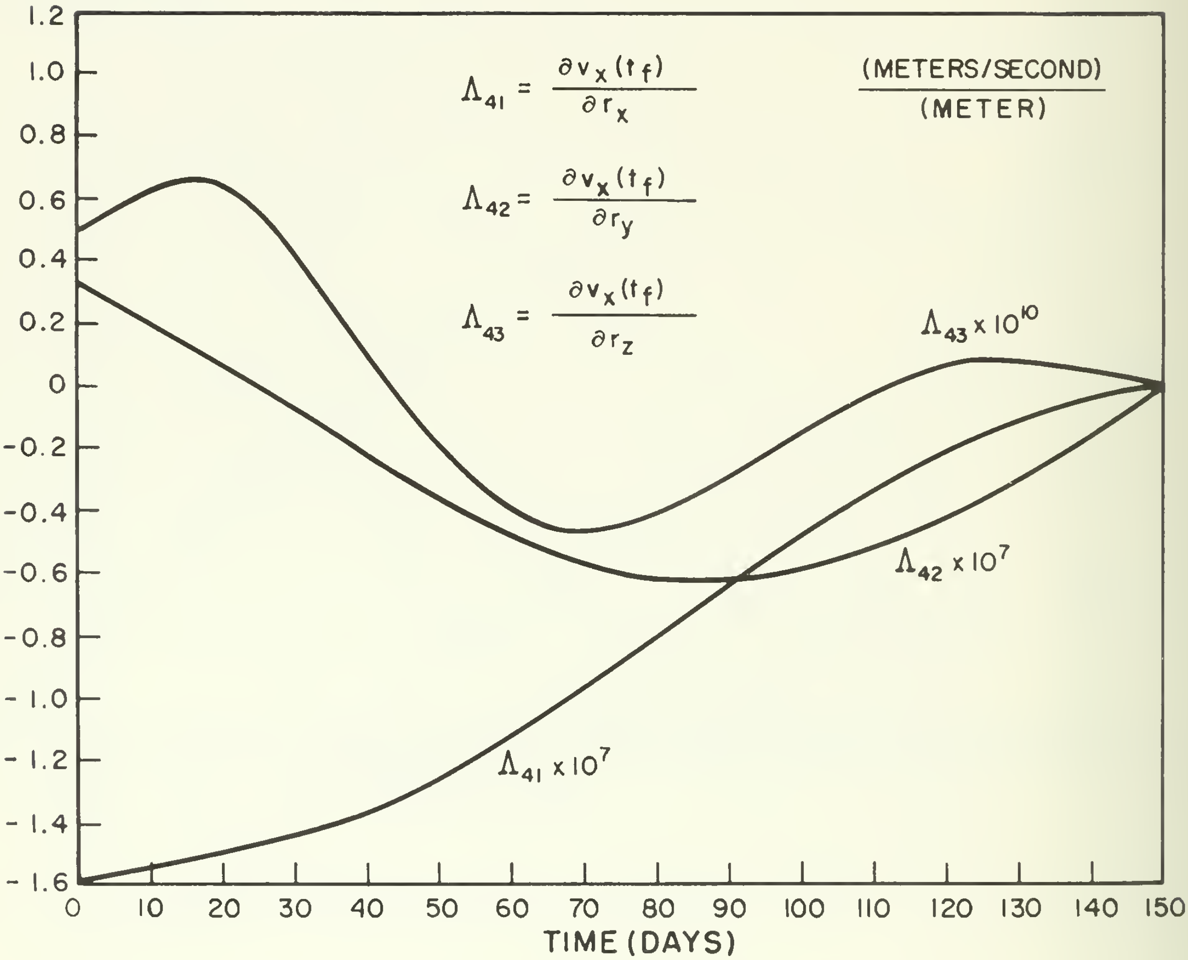

Fig. H-j. Sensitivity of $v_x(t_f)$ to position variations.

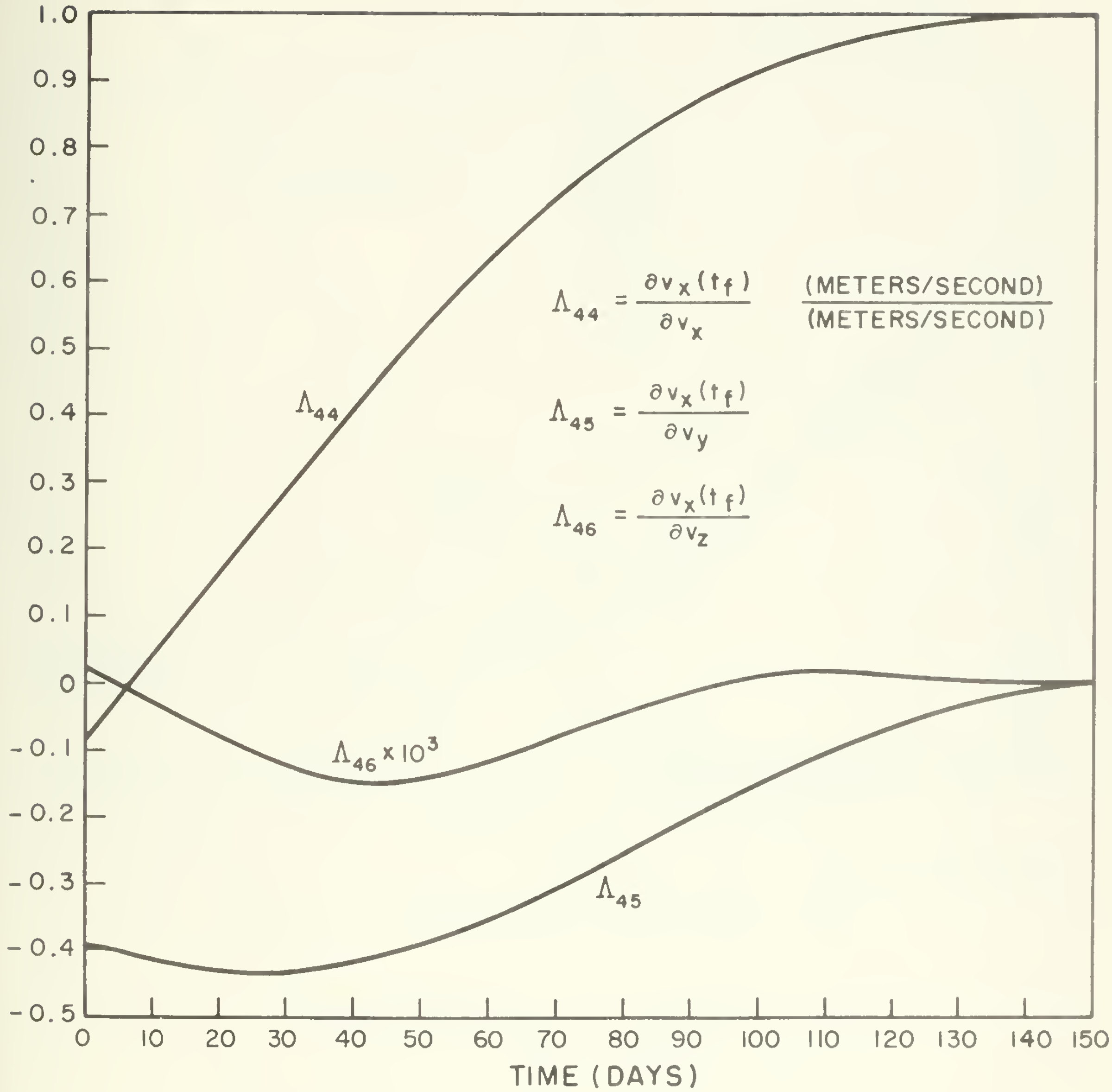

Fig. H-k. Sensitivity of v_x (t_f) to velocity variations.

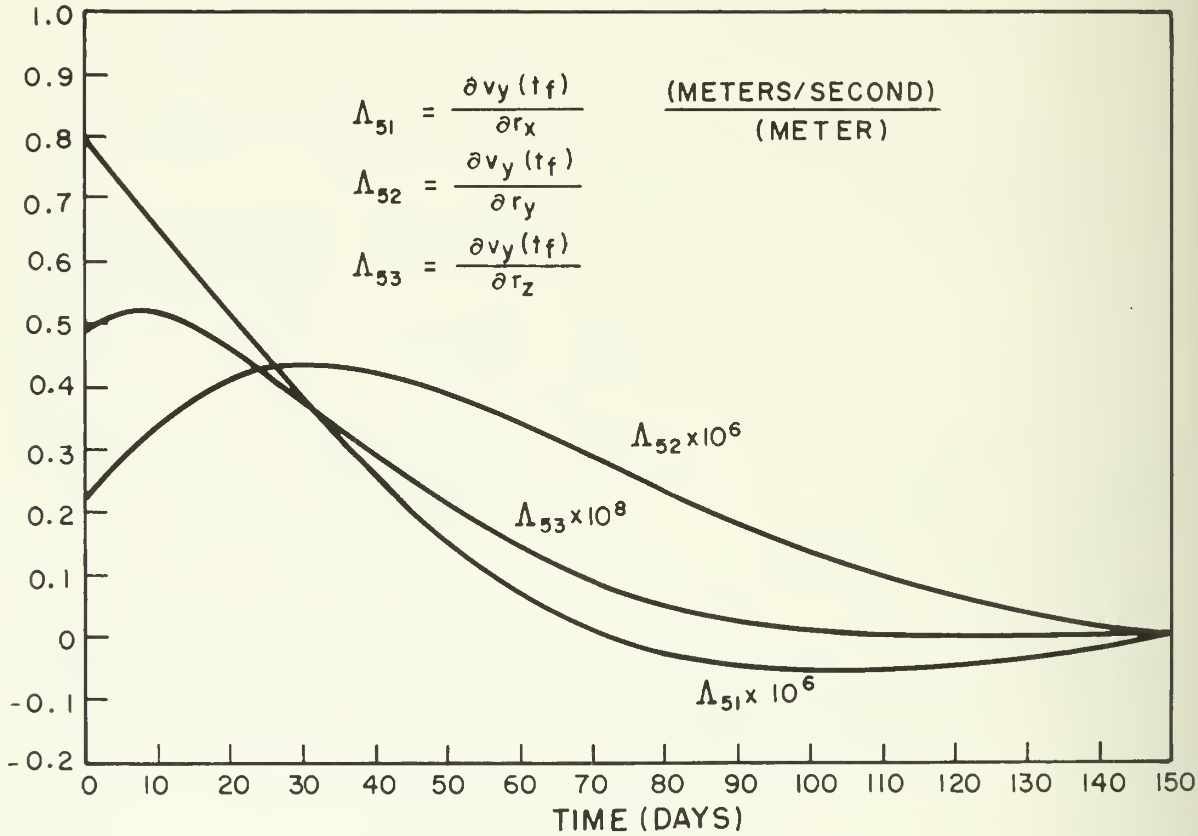

Fig. H-1. Sensitivity of v_y (t_f) to position variations.

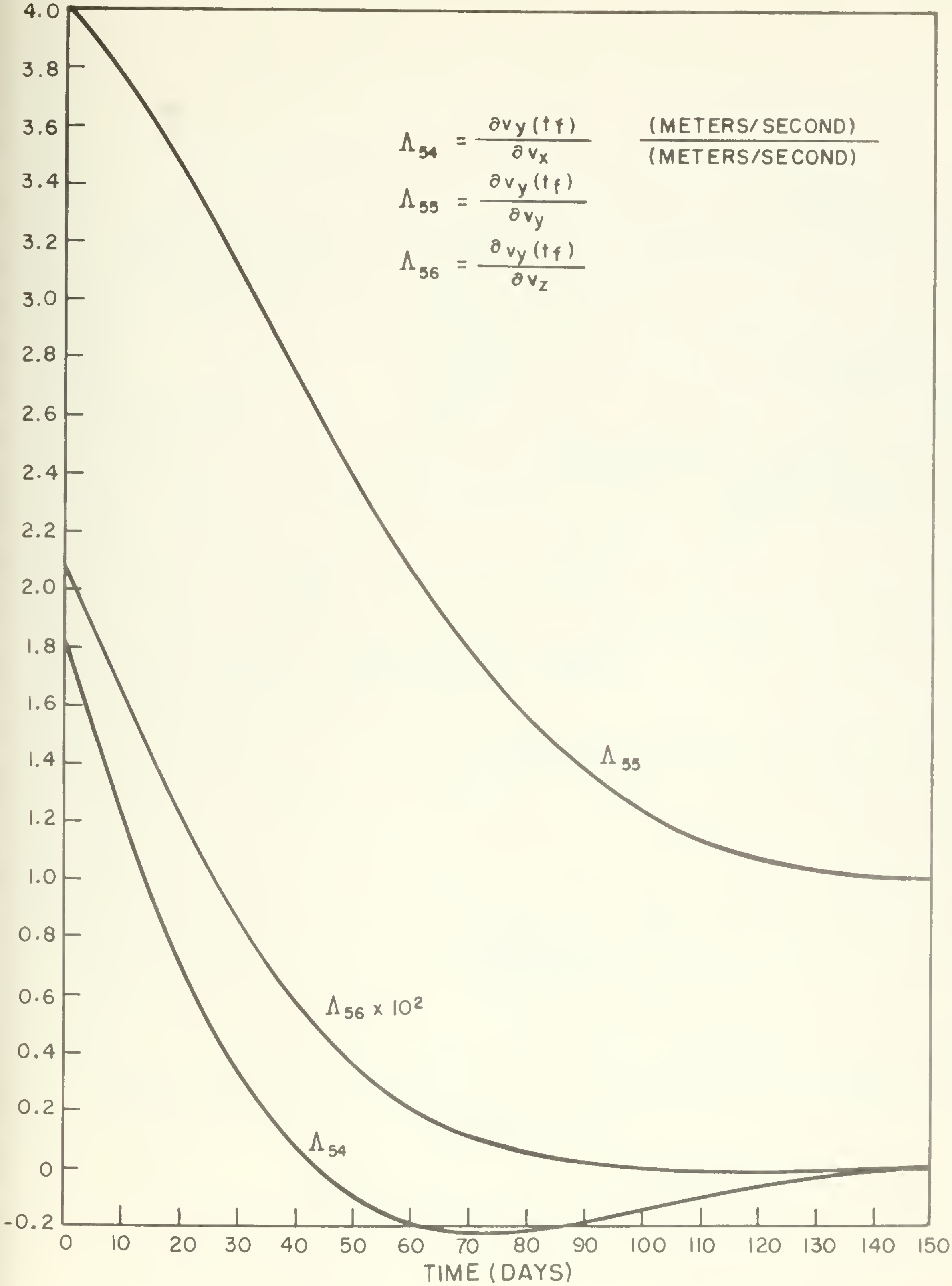

Fig. H-m. Sensitivity of v_g (t_f) to velocity variations.

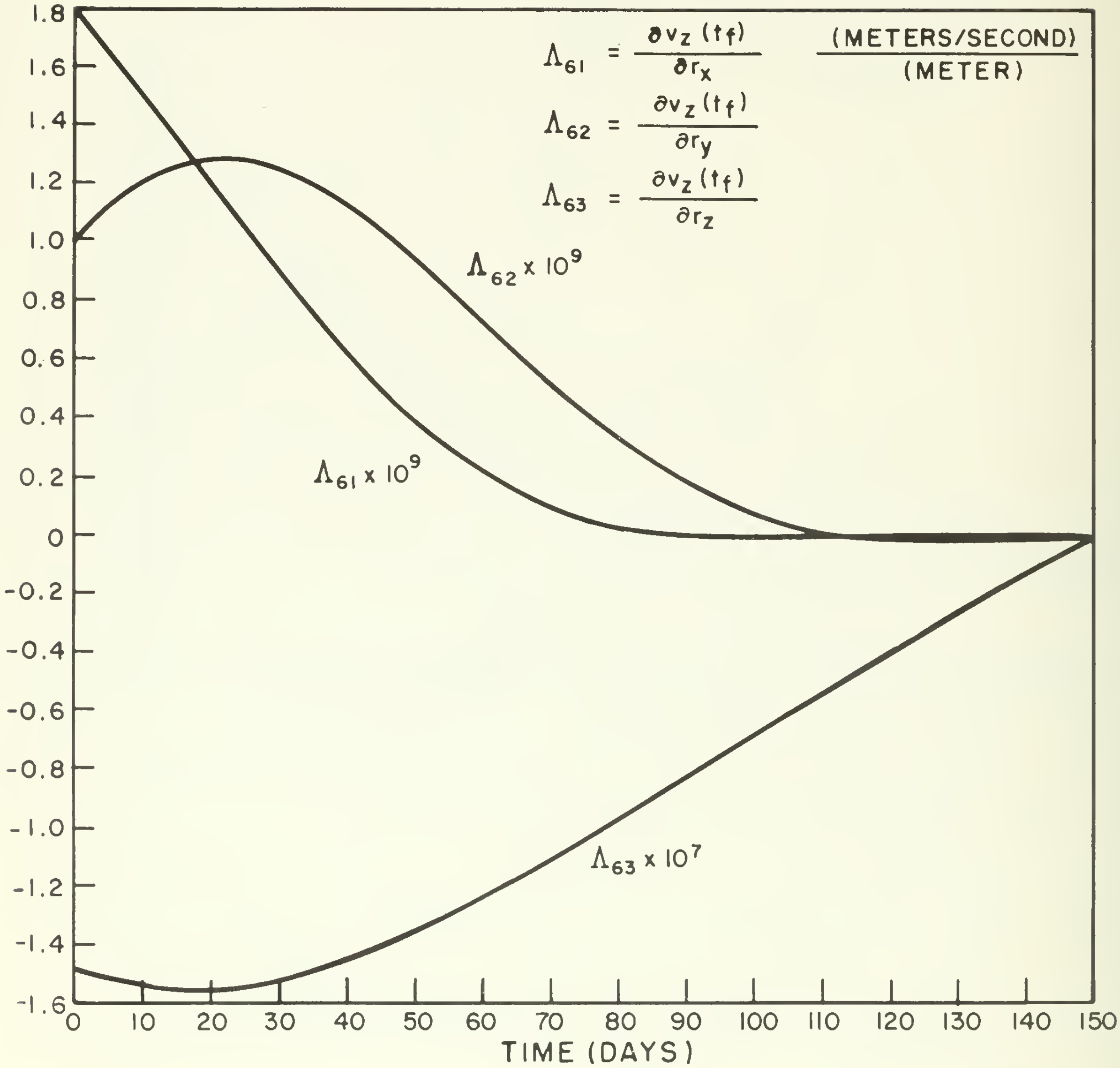

Fig. H-n. Sensitivity of v_z (t_f) to position variations.

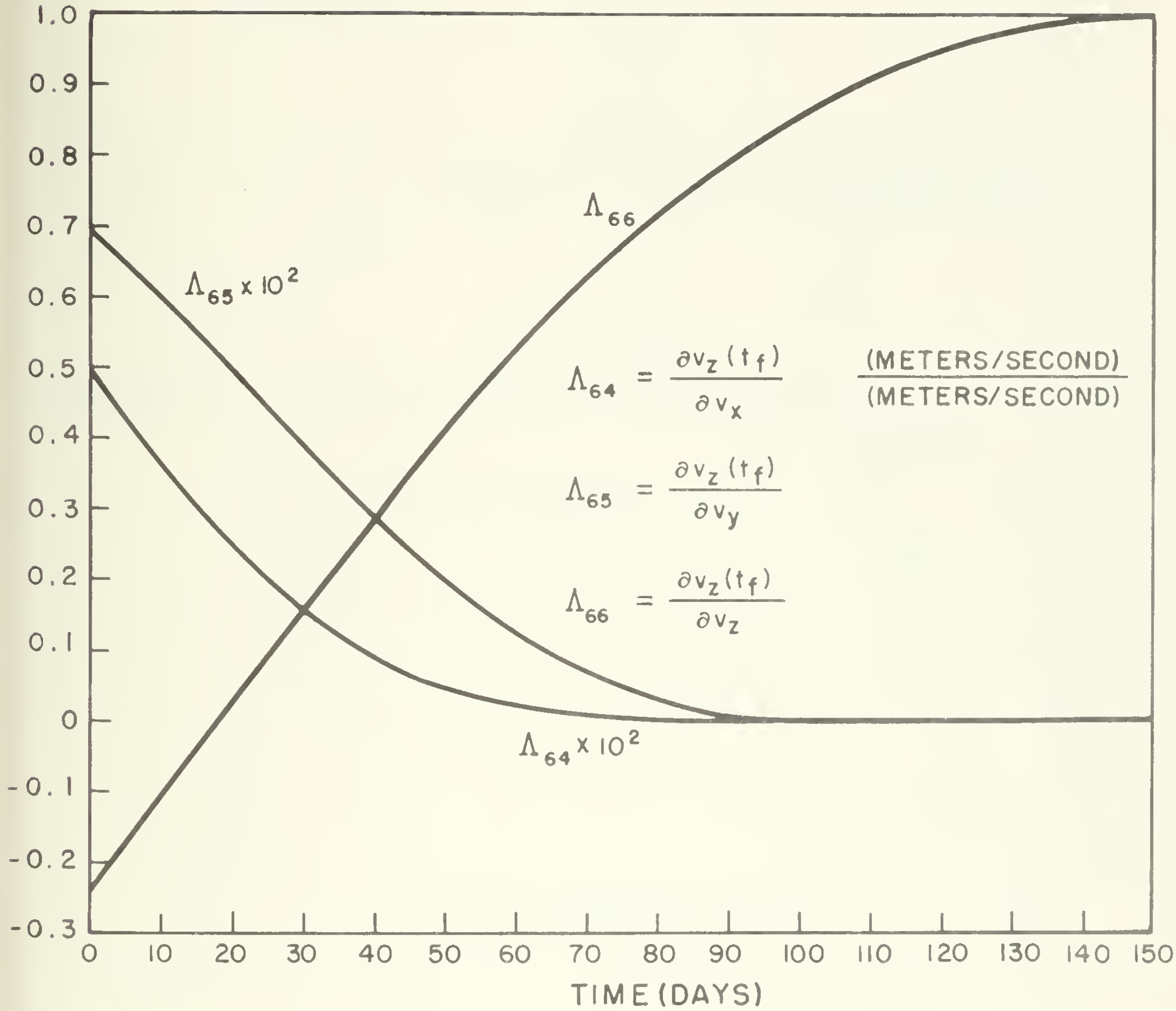

Fig. H-o. Sensitivity of v_z (t_f) to velocity variations.

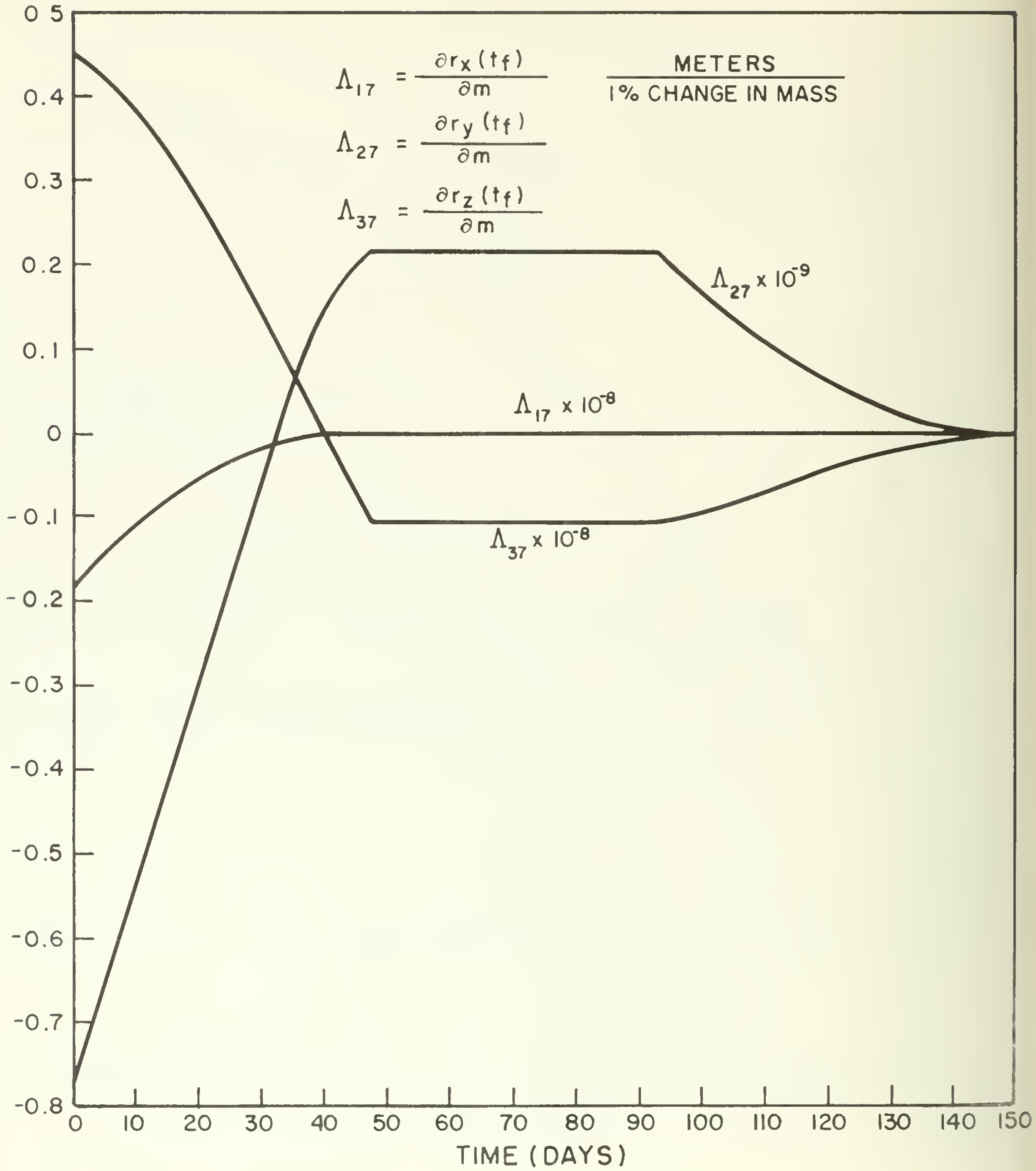

Fig. H-p. Sensitivity of r (t_f) to mass variation for optimal thrust program.

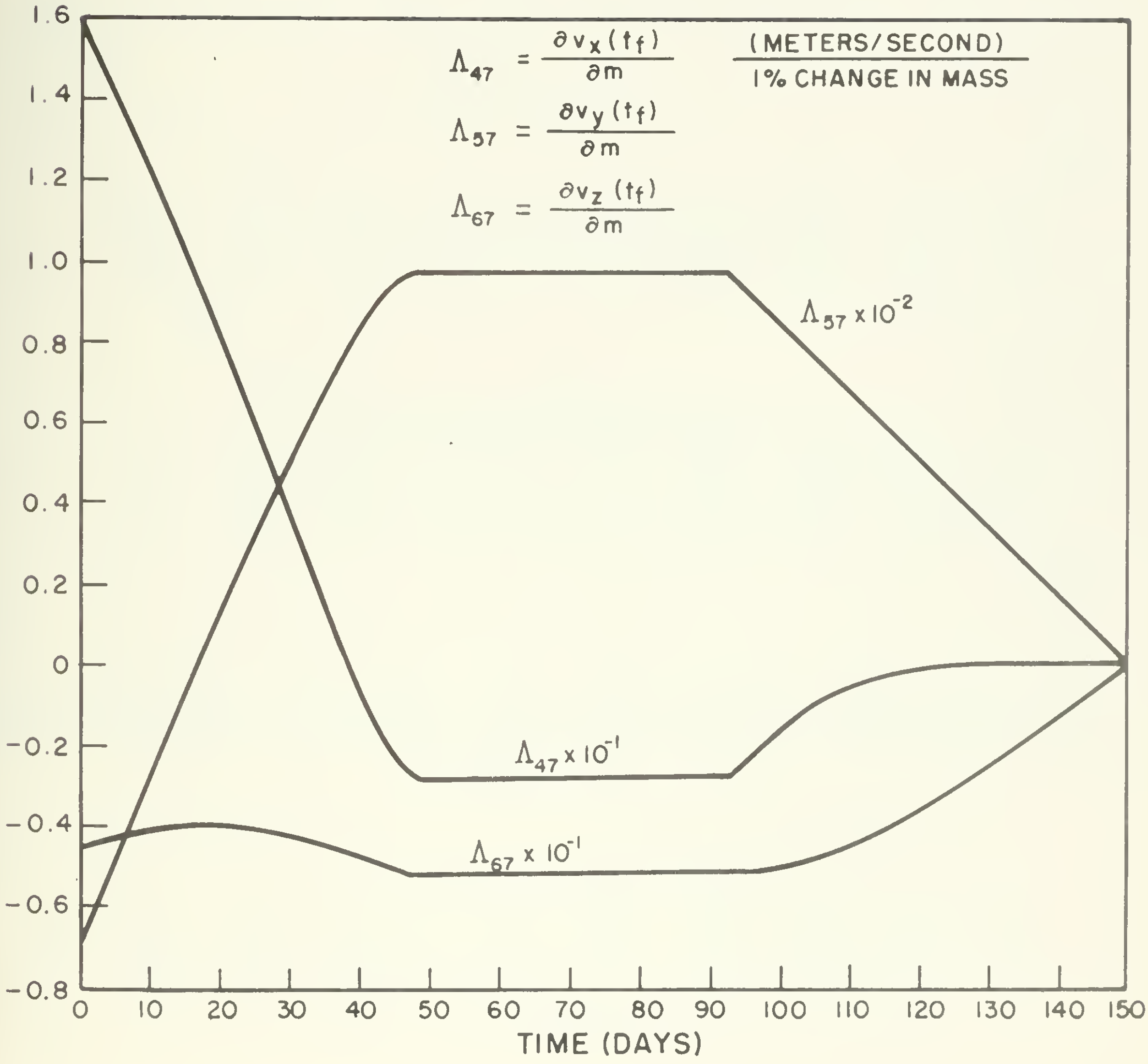

Fig. H-q. Sensitivity of v (t_f) to moss variation for optimal thrust program.

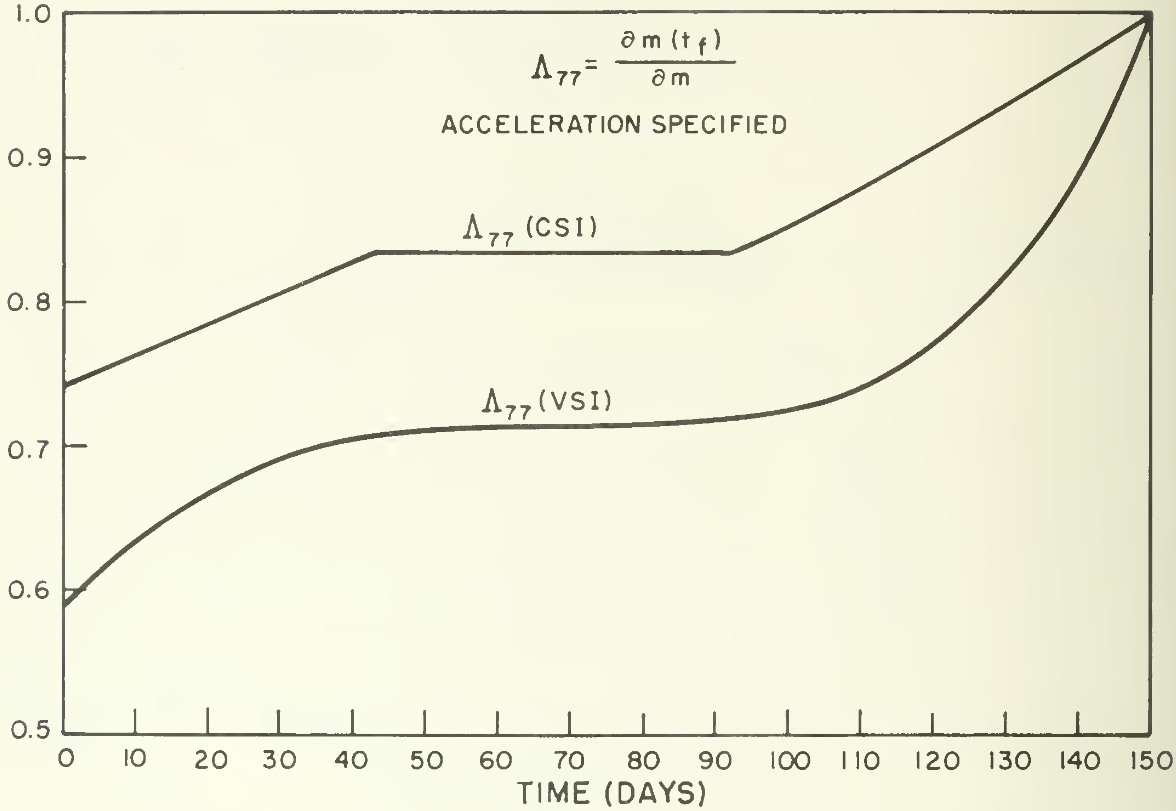

Fig. H-r. Sensitivity of $m(t_f)$ to mass variations for optimal acceleration program.

TIME	ACCELERATION X 10000/G	ACCEL MAG X 10000/G	POSITION	VELOCITY	MASS	GAMMA
0.	0.4358320	1.5736428	1.0116800	0.0002000	1.0000000	0.1573643
	1.4996772		0.	0.0169900		
	-0.1933153		0.	0.0004900		
4.0000	0.3711722	1.4911006	1.0103331	-0.0008748	0.9837995	0.1466944
	1.4314165		0.0684929	0.0172412		
	-0.1914657		0.0018822	0.0004508		
8.0000	0.3111782	1.4050215	1.0046762	-0.0019538	0.9697699	0.1362548
	1.3570681		0.1378064	0.0174000		
	-0.1887318		0.0036044	0.0004100		
12.0000	0.2563973	1.3160350	0.9947071	-0.0030295	0.9577135	0.1260384
	1.2774741		0.2075686	0.0174655		
	-0.1851177		0.0051609	0.0003680		
16.0000	0.2072166	1.2247785	0.9804552	-0.0040938	0.9474433	0.1160408
	1.1935305		0.2774058	0.0174377		
	-0.1806337		0.0065477	0.0003252		
20.0000	0.1638520	1.1318833	0.9619824	-0.0051386	0.9387823	0.1062592
	1.1061570		0.3469471	0.0173179		
	-0.1752968		0.0077623	0.0002820		
24.0000	0.1263445	1.0379582	0.9393837	-0.0061555	0.9315629	0.0966923
	1.0162622		0.4158295	0.0171087		
	-0.1691313		0.0088039	0.0002388		
28.0000	0.0945658	0.9435723	0.9127857	-0.0071369	0.9256263	0.0873395
	0.9247092		0.4837022	0.0168138		
	-0.1621690		0.0096733	0.0001960		
32.0000	0.0682319	0.8492377	0.8823451	-0.0080757	0.9208223	0.0781997
	0.8322828		0.5502317	0.0164379		
	-0.1544487		0.0103729	0.0001540		
36.0000	0.0469254	0.7553949	0.8482456	-0.0089654	0.9170092	0.0692704
	0.7396612		0.6151056	0.0159870		
	-0.1460160		0.0109066	0.0001131		

VARIABLE SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 2 of 13)

TIME	ACCELERATION X 1000/G	ACCEL. MAG X 1000/G	POSITION	VELOCITY	MASS	GAMMA
40.0000	[illegible]	0.6024011	0.9106342	-0.0098008	0.9140536	0.0605470
	[illegible]		0.6786381	0.0154674		
	[illegible]		[illegible]	0.0000738		
44.0000	[illegible]	[illegible]	[illegible]	-0.0101778	0.9118304	0.0520222
	[illegible]		[illegible]	0.0148862		
	[illegible]		[illegible]	0.0000364		
48.0000	[illegible]	[illegible]	0.7261541	-0.0112932	0.9102225	0.0436863
	[illegible]		0.7970532	0.0142596		
	0.1175799		0.0115742	0.0000012		
52.0000	[illegible]	0.3908053	0.6796556	-0.0119454	0.9091200	0.0355289
	[illegible]		0.8527041	0.0135679		
	[illegible]		0.0115123	-0.0000317		
56.0000	[illegible]	0.3032145	0.6305762	-0.0125336	0.9084199	0.0275446
	[illegible]		0.9055431	0.0128452		
	[illegible]		0.0113244	-0.0000618		
60.0000	-0.0079461	[illegible]	0.5794713	-0.0130382	0.9080250	0.0197526
	[illegible]		0.9554214	0.0120891		
	[illegible]		0.0110213	-0.0000892		
64.0000	-0.0105982	0.1353266	[illegible]	-0.0135204	0.9078428	0.0122855
	0.1142298		1.0022193	0.0113057		
	-0.0717811		[illegible]	-0.0001136		
68.0000	-0.0127729	0.0672886	[illegible]	-0.0139219	0.9077841	0.0061083
	0.0282359		1.0458384	0.0105006		
	-0.0597270		0.0101166	-0.0001349		
72.0000	-0.0148248	0.0761562	0.4149956	-0.0142654	0.9077615	0.0069141
	-0.0576785		1.0862018	0.0096786		
	-0.0474837		0.0095396	-0.0001531		
76.0000	-0.0170161	0.1491163	0.3573396	-0.0145536	0.9076879	0.0135351
	-0.1439237		1.1232499	0.0088436		
	-0.0351014		0.0088962	-0.0001681		

VARIABLE SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 3 of 13)

TIME	ACCELERATION X 10000/G	ACCEL MAG X 10000/G	POSITION	VELOCITY	MASS	GAMMA
80.0000	-0.0195260	0.2328575	0.2986361	-0.0147897	0.9074748	0.0211312
	-0.2309315		1.1569380	0.0079991		
	-0.0226267		0.0081993	-0.0001798		
84.0000	-0.0224512	0.3200893	0.2390873	-0.0149768	0.9070310	0.0290331
	-0.3191411		1.1872333	0.0071476		
	-0.0101019		0.0074618	-0.0001884		
88.0000	-0.0258117	0.4098072	0.1788824	-0.0151183	0.9062607	0.0371392
	-0.4089863		1.2141122	0.0062911		
	0.0024345		0.0066964	-0.0001938		
92.0000	-0.0295583	0.5019806	0.1181978	-0.0152172	0.9050630	0.0454324
	-0.5008866		1.2375576	0.0054311		
	0.0149478		0.0059157	-0.0001960		
96.0000	-0.0335799	0.5968157	0.0571976	-0.0152766	0.9033307	0.0539122
	-0.5952396		1.2575572	0.0045683		
	0.0274070		0.0051322	-0.0001952		
100.0000	-0.0377106	0.6945832	-0.0039658	-0.0152992	0.9009500	0.0625785
	-0.6924167		1.2741012	0.0037033		
	0.0397840		0.0043583	-0.0001913		
104.0000	-0.0417368	0.7955619	-0.0651509	-0.0152878	0.8978006	0.0714256
	-0.7927593		1.2871804	0.0028359		
	0.0520539		0.0036058	-0.0001844		
108.0000	-0.0454028	0.9000182	-0.1262256	-0.0152444	0.8937565	0.0804397
	-0.8965770		1.2967852	0.0019660		
	0.0641943		0.0028866	-0.0001747		
112.0000	-0.0484166	1.0081965	-0.1870667	-0.0151713	0.8886872	0.0895971
	-1.0041474		1.3029040	0.0010928		
	0.0761847		0.0022120	-0.0001621		
116.0000	-0.0504539	1.1203171	-0.2475582	-0.0150699	0.8824604	0.0988635
	-1.1157149		1.3055226	0.0002157		
	0.0880068		0.0015931	-0.0001469		

VARIABLE SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 4 of 13)

TIME	ACCELERATION X 1000076	ACCEL MAG X 1000076	POSITION	VELOCITY	MASS	GAMMA
120.0000	-0.0511616	1.2365751	-0.3075906	-0.0149419	0.8749442	0.1081934
	-1.2314917		1.3046229	-0.0006664		
	0.0996434		0.0010406	-0.0001289		
124.0000	-0.0501604	1.3571414	-0.3670590	-0.0147882	0.8660119	0.1175301
	-1.3516576		1.3001829	-0.0015546		
	0.1110788		0.0005649	-0.0001085		
128.0000	-0.0470466	1.4821645	-0.4258626	-0.0146096	0.8555457	0.1268059
	-1.4763609		1.2921760	-0.0024501		
	0.1222979		0.0001761	-0.0000855		
132.0000	-0.0413932	1.6117717	-0.4839030	-0.0144066	0.8434424	0.1359437
	-1.6057177		1.2805705	-0.0033541		
	0.1332861		-0.0001163	-0.0000603		
136.0000	-0.0327505	1.7460706	-0.5410831	-0.0141794	0.8296185	0.1448573
	-1.7398120		1.2653300	-0.0042679		
	0.1440289		-0.0003029	-0.0000327		
139.9999	-0.0206461	1.8851505	-0.5973062	-0.0139280	0.8140164	0.1534543
	-1.8786943		1.2464130	-0.0051926		
	0.1545120		-0.0003751	-0.0000030		
143.9999	-0.0045847	2.0290028	-0.6524743	-0.0136519	0.7966090	0.1616386
	-2.0223896		1.2237731	-0.0061295		
	0.1647205		-0.0003242	0.0000288		
147.9999	0.0159524	2.1779209	-0.7064876	-0.0133505	0.7774048	0.1693126
	-2.1708492		1.1973595	-0.0070796		
	0.1746387		-0.0001424	0.0000625		
150.0000	0.0280681	2.2541935	-0.7330300	-0.0131900	0.7671417	0.1729286
	-2.2468614		1.1827200	-0.0075600		
	0.1794839		-0.0000000	0.0000800		

INITIAL STATE	FINAL STATE	TARGET STATE	ETA	TARGET XI	INITIAL XI	NU X 10000
1.011679992	-0.733029984	-0.733029999	0.407624744	0.000000015	0.000000149	0.008260526
0.	1.182719991	1.182719991	0.739010081	0.	0.000000082	0.008035623
0.	-0.000000000	0.	0.000854947	-0.000000000	0.000000007	-0.000174090
0.000199996	-0.013189994	-0.013189994	-0.013253100	-0.000000000	-0.000000000	0.215666905
0.016989999	-0.007560000	-0.007560000	-0.003057636	-0.000000000	-0.000000000	0.742100216
0.000489995	0.000079997	0.000079997	-0.000670545	-0.000000000	0.000000000	-0.095660135

J	MSTAR MATRIX					
0.2121135E-06	2091619.593750	1688908.578125	21796.786865	-52980.811035	-20725.826904	-392.695793
	1688908.578125	2623689.843750	24848.495850	-50838.145508	-22524.258301	-413.146454
	21796.786865	24848.495850	426329.886719	-649.748077	-277.836868	-515.949066
	-52980.811035	-50838.145508	-649.748077	1425.978836	548.414879	10.385288
	-20725.826904	-22524.258301	-277.836868	548.414879	274.492645	4.159672
	-392.695793	-413.146454	-515.949066	10.385288	4.159672	58.103417

TIME	LAMBDA MATRIX						
0.	2.89224660	1.50126831	0.02722765	149.57493210	60.47617102	0.99680725	-0.12522033
	4.95882154	1.76201375	0.04373664	146.14898109	283.46575165	2.26823884	-0.53734816
	0.04924151	0.02470049	-0.75837098	1.41507199	1.51026335	64.01908207	0.03026927
	-0.01324889	0.00313156	0.00001214	-0.07587107	-0.38159077	0.00010534	0.00096298
	0.06868354	0.01856754	0.00060896	1.87054859	3.98716348	0.03477781	-0.00445037
	0.00029206	0.00016309	-0.01285656	0.00939358	0.01326356	-0.23313799	-0.00026500
	0.	0.	0.	0.	0.	0.	1.00000000
							0.58850709
4.00	2.55765384	1.55013624	0.02783629	138.68027496	54.36001778	0.88633537	-0.09685020
	4.61580720	2.06452864	0.04575456	126.99403763	275.79822922	2.08887774	-0.44457886
	0.04585796	0.02620141	-0.68361872	1.22477730	1.40830794	66.90408325	0.02828708
	-0.01309113	0.00269586	0.00001242	-0.02320108	-0.39324579	0.00005630	0.00084555
	0.06427441	0.02235781	0.00064122	1.60454684	3.90413302	0.03227250	-0.00315674
	0.00027112	0.00017682	-0.01309320	0.00826754	0.01258278	-0.18121850	-0.00025254
	0.	0.	0.	0.	0.	0.	1.00000000
							0.60804887
8.00	2.24057493	1.56059696	0.02750182	120.09033012	48.12631798	0.77537433	-0.07314450
	4.25804782	2.32307112	0.04668086	109.24220371	267.00817108	1.90365836	-0.35974347
	0.04221907	0.02723338	-0.60602728	1.04854925	1.30127102	69.48424435	0.02611936
	-0.01295970	0.00225537	0.00001279	0.02889317	-0.40315118	0.00000584	0.00073311
	0.05963341	0.02659748	0.00065917	1.35666274	3.80503526	0.02966712	-0.00196559
	0.00025099	0.00018777	-0.01326951	0.00722353	0.01185269	-0.12847283	-0.00024258
	0.	0.	0.	0.	0.	0.	1.00000000
							0.62576931
12.00	1.94494012	1.53628838	0.02640099	120.72705936	41.92162895	0.66734243	-0.05371151
	3.89048618	2.53668079	0.04661218	92.94273949	257.27359772	1.71675809	-0.28294301
	0.03839560	0.02779989	-0.52610253	0.88727121	1.19103536	71.74919796	0.02379969
	-0.01284596	0.00180287	0.00001299	0.08050018	-0.41127265	-0.00004582	0.00062601
	0.05481669	0.02976742	0.00066390	1.12771465	3.69211304	0.02701677	-0.00087930
	0.00023136	0.00019605	-0.01338492	0.00625894	0.01108415	-0.07514361	-0.00023518
	0.	0.	0.	0.	0.	0.	1.00000000
							0.64162380
16.00	1.67421873	1.48143876	0.02470721	113.49755955	35.87675619	0.56495669	-0.03811161
	3.51848915	2.70512837	0.04565839	78.12423515	246.77497482	1.53194158	-0.21413611
	0.03446716	0.02788735	-0.44438969	0.74152458	1.07950456	73.69068050	0.02136484
	-0.01274017	0.00133339	0.00001282	0.13167150	-0.41755141	-0.00009760	0.00052461
	0.04988732	0.03235605	0.00065668	0.91827956	3.56767130	0.02437183	0.00010157
	0.00021198	0.00020171	-0.01343942	0.00537232	0.01028772	-0.02147467	-0.00023030
	0.	0.	0.	0.	0.	0.	1.00000000
							0.65560947

TIME	LAMBDA MATRIX						
20.00	1.43123320	1.40078624	0.02258699	107.29628754	30.10453987	0.47025287	-0.02587724
	3.14764813	2.82905293	0.04394189	64.79328251	235.69197655	1.35250835	-0.15315237
	0.03051810	0.02751701	-0.36148307	0.61156138	0.96854760	75.30272102	0.01885363
	-0.01263191	0.00084437	0.00001219	0.18241823	-0.42191358	-0.00014780	0.00042932
	0.04491320	0.03436146	0.00063892	0.72867518	3.43404269	0.02177736	0.00097819
	0.00019277	0.00020476	-0.01343364	0.00456283	0.00947389	0.03229137	-0.00022793
	0.	0.	0.	0.	0.	0.	1.00000000
							0.66776221
24.00	1.21800505	1.29943968	0.02019581	102.00797272	24.69806767	0.38462137	-0.01653218
	2.78354448	2.91003686	0.04159514	52.93405247	224.19984818	1.18124712	-0.09970950
	0.02663270	0.02672049	-0.27800234	0.49729453	0.85993742	76.58177757	0.01630572
	-0.01251069	0.00033589	0.00001106	0.23270945	-0.42428032	-0.00019448	0.00034056
	0.03996446	0.03579293	0.00061215	0.55893998	3.29354537	0.01927249	0.00175325
	0.00017368	0.00020523	-0.01336891	0.00382998	0.00865303	0.08591582	-0.00022791
	0.	0.	0.	0.	0.	0.	1.00000000
							0.67815229
28.00	1.03564823	1.18269344	0.01767379	97.51104164	19.72955513	0.30885997	-0.00960912
	2.43149704	2.95060655	0.03875671	42.50882864	212.46560287	1.02040289	-0.05343293
	0.02289077	0.02554400	-0.19457808	0.39830755	0.75529090	77.52681351	0.01376048
	-0.01236651	-0.00018950	0.00000948	0.28247307	-0.42457804	-0.00023571	0.00025873
	0.03511039	0.03667153	0.00057803	0.40883311	3.14843687	0.01688994	0.00243071
	0.00015481	0.00020315	-0.01324730	0.00317308	0.00783540	0.13916679	-0.00023010
	0.	0.	0.	0.	0.	0.	1.00000000
							0.68687898
32.00	0.88432546	1.05581461	0.01514193	93.68136501	15.25002015	0.24324314	-0.00466479
	2.09631902	2.95415714	0.03556618	33.45955276	200.64438248	0.87166271	-0.01387691
	0.01936226	0.02404609	-0.11183389	0.31388301	0.65601435	78.13930798	0.01125586
	-0.01219050	-0.00072692	0.00000753	0.33159905	-0.42274816	-0.00026985	0.00018420
	0.03041641	0.03702977	0.00053827	0.27784308	3.00086740	0.01465572	0.00301557
	0.00013630	0.00019865	-0.01307161	0.00259099	0.00703099	0.19182213	-0.00023426
	0.	0.	0.	0.	0.	0.	1.00000000
							0.69406468
36.00	0.76327201	0.92382623	0.01269934	90.39603996	11.28983164	0.18760445	-0.00129186
	1.78210807	2.92480898	0.03215865	25.71028209	188.87623596	0.73616230	0.01945426
	0.01610456	0.02229425	-0.03036830	0.24304762	0.56326091	78.42318916	0.00882739
	-0.01197539	-0.00126961	0.00000532	0.37994487	-0.41875553	-0.00029563	0.00011730
	0.02594118	0.03691045	0.00049455	0.16520973	2.85283586	0.01258900	0.00351365
	0.00011837	0.00019185	-0.01284523	0.00208186	0.00624926	0.24367209	-0.00024015
	0.	0.	0.	0.	0.	0.	1.00000000
							0.69984882

TIME	LAMBDA MATRIX						
40.00	0.67088475	0.79130920	0.01042197	87.53693485	7.86008954	0.14142706	0.00087246
	1.49209133	2.86721116	0.02865945	19.17038631	177.28359032	0.61451501	0.04708585
	0.01315960	0.02036086	0.04926199	0.18462878	0.47790241	78.38470268	0.00650740
	-0.01171590	-0.00180938	0.00000300	0.42734289	-0.41259509	-0.00031231	0.00005827
	0.02173430	0.03636471	0.00044850	0.06995537	2.70615256	0.01070235	0.00393123
	0.00010124	0.00018300	-0.01257215	0.00164293	0.00549889	0.29452173	-0.00024748
	0.	0.	0.	0.	0.	0.	1.00000000
							0.70438199
44.00	0.60486335	0.66224259	0.00836264	84.99378109	4.95472670	0.10393696	0.00214146
	1.22853832	2.78631514	0.02517961	13.73821998	165.96961021	0.50686005	0.06955422
	0.01055297	0.01831882	0.12655628	0.13731892	0.40051872	78.03220940	0.00432433
	-0.01140902	-0.00233728	0.00000069	0.47360882	-0.40429624	-0.00031967	0.00000724
	0.01783472	0.03544950	0.00040158	-0.00907497	2.56241086	0.00900211	0.00427490
	0.00008515	0.00017238	-0.01225673	0.00127052	0.00478759	0.34419284	-0.00025594
	0.	0.	0.	0.	0.	0.	1.00000000
							0.70782088
48.00	0.56238685	0.53989562	0.00655268	82.66661835	2.55315465	0.07419255	0.00277695
	0.99274286	2.68714195	0.02181262	9.30503011	155.01743126	0.41292675	0.08739084
	0.00829429	0.01623758	0.20108084	0.09974031	0.33140375	77.37594604	0.00230231
	-0.01105395	-0.00284415	-0.00000148	0.51855074	-0.39392485	-0.00031804	-0.00003575
	0.01426988	0.03422491	0.00035510	-0.07316931	2.42296901	0.00748906	0.00455128
	0.00007030	0.00016031	-0.01190365	0.00096007	0.00412177	0.39252538	-0.00026522
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71032384
52.00	0.54030602	0.42677690	0.00500451	80.46748257	0.62322421	0.05116571	0.00299022
	0.78507274	2.57456651	0.01863257	5.75873625	144.49032974	0.33210891	0.10110939
	0.00637873	0.01417961	0.27247563	0.07050646	0.27058662	76.42773724	0.00046083
	-0.01065204	-0.00332118	-0.00000340	0.56197792	-0.38158284	-0.00030818	-0.00007081
	0.01105583	0.03275143	0.00031015	-0.12370227	2.28894326	0.00615922	0.00476685
	0.00005685	0.00014718	-0.01151767	0.00070626	0.00350647	0.43937821	-0.00027497
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71204772
56.00	0.53533041	0.32463735	0.00371538	78.32136154	-0.87572926	0.03381088	0.00294522
	0.60507064	2.45313564	0.01569380	2.98752287	134.43268776	0.26354356	0.11119591
	0.00478947	0.01219801	0.34045725	0.04827496	0.21786455	75.20069408	-0.00118528
	-0.01020646	-0.00376040	-0.00000501	0.60370877	-0.36740585	-0.00029121	-0.00009817
	0.00819792	0.03108755	0.00026757	-0.16209120	2.16121119	0.00500470	0.00492781
	0.00004492	0.00013335	-0.01110359	0.00050324	0.00294523	0.48462933	-0.00028487
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71314564

VARIABLE SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 9 of 13)

TIME	LAMBDA MATRIX						
60.00	0.54419596	0.23452080	0.00266958	76.16639042	-1.98993276	0.02111944	0.00276335
	0.45159215	2.32693267	0.01303179	0.88278603	124.87158489	0.20618871	0.11810208
	0.00350055	0.01033504	0.40481813	0.03178995	0.17284386	73.70890903	-0.00262589
	-0.00972193	-0.00415498	-0.00000627	0.64357764	-0.35155920	-0.00026852	-0.00011818
	0.00569208	0.02928782	0.00022800	-0.18975540	2.04042366	0.00401465	0.00503993
	0.00003456	0.00011920	-0.01066601	0.00034481	0.00244006	0.52817563	-0.00029457
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71376603
64.00	0.56380099	0.15684674	0.00184514	73.95348454	-2.76851031	0.01215925	0.00252920
	0.32296159	2.19949016	0.01066510	-0.65836015	115.81884956	0.15889478	0.12224127
	0.00247991	0.00862160	0.46542238	0.01991273	0.13498439	71.96715927	-0.00385531
	-0.00920422	-0.00449947	-0.00000713	0.68143996	-0.33423282	-0.00024159	-0.00013132
	0.00352649	0.02740135	0.00019185	-0.20808186	1.92702396	0.00317612	0.00510851
	0.00002576	0.00010508	-0.01020935	0.00022466	0.00199155	0.56993203	-0.00030380
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71405253
68.00	0.59130712	0.09151512	0.00121412	71.64546204	-3.26118195	0.00610033	0.00229689
	0.21713026	2.07375067	0.00859792	-1.73131469	107.27336311	0.12046792	0.12398705
	0.00169224	0.00707776	0.52219949	0.01164003	0.10364442	69.99063396	-0.00487205
	-0.00865979	-0.00478984	-0.00000762	0.71717577	-0.31563562	-0.00021194	-0.00013819
	0.00168332	0.02547098	0.00015936	-0.21839736	1.82127109	0.00247493	0.00513826
	0.00001848	0.00009129	-0.00973766	0.00013664	0.00159895	0.60983044	-0.00031222
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71414486
72.00	0.62420344	0.03801730	0.00074694	69.21585941	-3.51641926	0.00222785	0.00209633
	0.13182360	1.95206724	0.00682288	-2.42277715	99.22341156	0.08972149	0.12367368
	0.00110140	0.00571364	0.57513619	0.00611236	0.07812228	67.79468536	-0.00567847
	-0.00809544	-0.00502348	-0.00000776	0.75069177	-0.29598977	-0.00018104	-0.00013945
	0.00014052	0.02353290	0.00013058	-0.22194900	1.72326607	0.00189627	0.00513337
	0.00001262	0.00007811	-0.00925464	0.00007486	0.00126036	0.64781827	-0.00031957
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71418038
76.00	0.66033815	-0.00445359	0.00041427	66.64752674	-3.58002964	-0.00005454	0.00193922
	0.06466950	1.83623561	0.00532401	-2.81011131	91.64900589	0.06551630	0.12159807
	0.00067241	0.00453097	0.62426765	0.00261299	0.05769305	65.39461994	-0.00628038
	-0.00751796	-0.00519906	-0.00000759	0.78192184	-0.27552523	-0.00015022	-0.00013585
	-0.00112657	0.02161669	0.00010546	-0.21988939	1.63297844	0.00142536	0.00509741
	0.00000865	0.00006575	-0.00876356	0.00003392	0.00097291	0.68385686	-0.00032558
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71429617

LAMBDA MATRIX

TIME							
80.00	0.69792383	-0.03690348	0.00018855	63.93122959	-3.49415880	-0.00122888	0.00182442
	0.01330292	1.72754779	0.00407936	-2.96117526	84.52398109	0.04679022	0.11802296
	0.00037283	0.00352465	0.66966832	0.00056060	0.04163910	62.80551958	-0.00668664
	-0.00693386	-0.00531640	-0.00000716	0.81082664	-0.25447495	-0.00012061	-0.00012822
	-0.00214387	0.01974569	0.00008383	-0.21326951	1.55027194	0.00104789	0.00503339
	0.00000460	0.00005437	-0.00826729	0.00000897	0.00073301	0.71791987	-0.00033002
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71463162
84.00	0.73552209	-0.06045289	0.00004507	61.06413746	-3.29667446	-0.00167245	0.00174264
	-0.02455575	1.62685804	0.00306336	-2.93454081	77.81793785	0.03257646	0.11318063
	0.00017370	0.00268449	0.71144319	-0.00050337	0.02927413	60.04210758	-0.00690873
	-0.00634928	-0.00537623	-0.00000653	0.83739213	-0.23307070	-0.00009315	-0.00011741
	-0.00293734	0.01793759	0.00006543	-0.20303677	1.47492877	0.00075031	0.00494377
	0.00000214	0.00004407	-0.00776827	-0.00000421	0.00053650	0.74999157	-0.00033268
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71533123
88.00	0.77201496	-0.07626717	-0.00003754	58.04854012	-3.02084622	-0.00167038	0.00168039
	-0.05104449	1.53465281	0.00224875	-2.77988758	71.49779510	0.02201474	0.10727718
	0.00004996	0.00199668	0.74971962	-0.00092931	0.01995999	57.11863852	-0.00696033
	-0.00576981	-0.00538003	-0.00000575	0.86162777	-0.21153978	-0.00006854	-0.00010425
	-0.00353219	0.01620514	0.00005011	-0.19003561	1.40667017	0.00052008	0.00483049
	0.00000048	0.00003493	-0.00726856	-0.00000921	0.00037888	0.78006516	-0.00033335
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71654775
92.00	0.80656806	-0.08550460	-0.00007715	54.89061117	-2.69530028	-0.00142923	0.00162301
	-0.06812524	1.45112132	0.00160807	-2.53871030	65.52914238	0.01435465	0.10049692
	-0.00001933	0.00144524	0.78464019	-0.00097553	0.01311879	54.04882479	-0.00685699
	-0.00520045	-0.00532980	-0.00000488	0.88356417	-0.19910248	-0.00004724	-0.00008961
	-0.00395224	0.01455691	0.00003745	-0.17501210	1.34517519	0.00034581	0.00469501
	-0.00000051	0.00002696	-0.00676981	-0.00000896	0.00025549	0.80814132	-0.00033186
	0.	0.	0.	0.	0.	0.	1.00000000
							0.71844543
96.00	0.83859096	-0.08927791	-0.00008815	51.59938002	-2.34409329	-0.00109112	0.00155713
	-0.07756651	1.37621962	0.00111482	-2.24506770	59.37730265	0.00895370	0.09300669
	-0.00005122	0.00101310	0.81635633	-0.00082430	0.00823893	50.84578943	-0.00661570
	-0.00464562	-0.00522789	-0.00000397	0.90325081	-0.16897029	-0.00002950	-0.00007429
	-0.00421954	0.01299801	0.00002717	-0.15862099	1.29009560	0.00021730	0.00453831
	-0.00000101	0.00002015	-0.00627336	-0.00000574	0.00016163	0.83422668	-0.00032807
	0.	0.	0.	0.	0.	0.	1.00000000
							0.72120352

TIME	LAMBDA MATRIX						
100.00	0.86769762	-0.08863013	-0.00008173	48.18577576	-1.98696744	-0.00074714	0.00147223
	-0.08093868	1.30972755	0.00074416	-1.92626847	54.50816536	0.00527263	0.08496018
	-0.00005905	0.00068394	0.84502321	-0.00059772	0.00487785	47.52203941	-0.00625466
	-0.00410918	-0.00507686	-0.00000308	0.92075374	-0.14834486	-0.00001539	-0.00005907
	-0.00435407	0.01153079	0.00001901	-0.14143272	1.24106869	0.00012558	0.00436094
	-0.00000114	0.00001446	-0.00578025	-0.00000133	0.00009275	0.85833266	-0.00032184
	0.	0.	0.	0.	0.	0.	1.00000000
							0.72501998
104.00	0.89366920	-0.08451929	-0.00006626	44.66197634	-1.63966802	-0.00044927	0.00136174
	-0.07961620	1.25129886	0.00047344	-1.60380973	49.38873672	0.00286728	0.07650201
	-0.00005299	0.00043837	0.87079539	-0.00037025	0.00266075	44.08946180	-0.00579293
	-0.00359446	-0.00487937	-0.00000223	0.93615326	-0.12841733	-0.00000479	-0.00004464
	-0.00437376	0.01015550	0.00001264	-0.12394149	1.19772656	0.00006282	0.00416305
	-0.00000102	0.00000983	-0.00529122	0.00000306	0.00004449	0.88047414	-0.00031306
	0.	0.	0.	0.	0.	0.	1.00000000
							0.73011545
108.00	0.91642068	-0.07781182	-0.00004768	41.04072380	-1.31426917	-0.00022113	0.00122350
	-0.07479661	1.20049959	0.00028247	-1.29399453	44.48760843	0.00137889	0.06777167
	-0.00004036	0.00026344	0.89382349	-0.00018236	0.00127797	40.55933285	-0.00525030
	-0.00310437	-0.00463804	-0.00000146	0.94954223	-0.10936831	0.00000257	-0.00003159
	-0.00429442	0.00887076	0.00000783	-0.10657453	1.15970394	0.00002233	0.00394440
	-0.00000075	0.00000618	-0.00480682	0.00000664	0.00001276	0.90066864	-0.00030161
	0.	0.	0.	0.	0.	0.	1.00000000
							0.73673773
112.00	0.94597254	-0.06928248	-0.00002991	37.33488798	-1.01959492	-0.00006669	0.00105981
	-0.06750557	1.15684149	0.00015358	-1.00872321	39.77521038	0.00052483	0.05890728
	-0.00002615	0.00014393	0.91425145	-0.00004935	0.00047940	36.94233894	-0.00464702
	-0.00264145	-0.00435540	-0.00000080	0.96102440	-0.09136805	0.00000708	-0.00002044
	-0.00412985	0.00767406	0.00000431	-0.08969963	1.12664314	-0.00000156	0.00370438
	-0.00000041	0.00000340	-0.00432734	0.00000898	-0.00000612	0.91893527	-0.00028743
	0.	0.	0.	0.	0.	0.	1.00000000
							0.74516668
116.00	0.95242658	-0.05961777	-0.00001523	33.55708313	-0.76152017	0.00002235	0.00087709
	-0.05862479	1.11980677	0.00007158	-0.75605055	35.22403765	0.00008784	0.05004941
	-0.00001348	0.00006710	0.93221466	0.00002922	0.00006968	33.24860620	-0.00400374
	-0.00220796	-0.00403386	-0.00000027	0.97071305	-0.07457694	0.00000920	-0.00001155
	-0.00389200	0.00656213	0.00000186	-0.07363319	1.09819838	-0.00001357	0.00344199
	-0.00000007	0.00000139	-0.00385295	0.00000996	-0.00001547	0.93529415	-0.00027040
	0.	0.	0.	0.	0.	0.	1.00000000
							0.75571997

LAMBDA MATRIX

TIME							
120.00	0.96594685	-0.04942168	-0.00000465	29.71938777	-0.54335173	0.00006066	0.00068517
	-0.04890257	1.03886582	0.00002369	-0.54083041	30.80863476	-0.00009315	0.04134469
	-0.00000393	0.00002185	0.94783847	0.00006281	-0.00009938	29.48773956	-0.00334144
	-0.00180598	-0.00367562	0.00000012	0.97873014	-0.05914608	0.00000946	-0.00000513
	-0.00359108	0.00553122	0.00000026	-0.05864757	1.07403804	-0.00001755	0.00315587
	0.00000020	0.00000006	-0.00338363	0.00000969	-0.00001820	0.94976563	-0.00025047
	0.	0.	0.	0.	0.	0.	1.00000000
							0.76875962
124.00	0.97674454	-0.03922367	0.00000175	25.83314276	-0.36614642	0.00006509	0.00049622
	-0.03897451	1.06349051	-0.00000060	-0.36510117	26.50568366	-0.00013302	0.03294984
	0.00000198	-0.00000124	0.96123717	0.00006565	-0.00013457	25.66886353	-0.00268140
	-0.00143746	-0.00328275	0.00000038	0.98520557	-0.04521807	0.00000841	-0.00000123
	-0.00323575	0.00457740	-0.00000064	-0.04497699	1.05384564	-0.00001660	0.00284425
	0.00000040	-0.00000071	-0.00291927	0.00000843	-0.00001673	0.96236981	-0.00022755
	0.	0.	0.	0.	0.	0.	1.00000000
							0.78469984
128.00	0.98506687	-0.02948680	0.00000456	21.90873981	-0.22893573	0.00005145	0.00032350
	-0.02938021	1.04316361	-0.00000976	-0.22857779	22.29399037	-0.00010861	0.02503572
	0.00000461	-0.00000992	0.97251327	0.00005138	-0.00010860	21.80067039	-0.00204527
	-0.00110432	-0.00285709	0.00000051	0.99027701	-0.03292774	0.00000659	0.00000030
	-0.00283330	0.00369666	-0.00000104	-0.03282421	1.03732143	-0.00001309	0.00250496
	0.00000051	-0.00000105	-0.00245965	0.00000656	-0.00001307	0.97312612	-0.00020157
	0.	0.	0.	0.	0.	0.	1.00000000
							0.80401625
132.00	0.99118903	-0.02061646	0.00000475	17.95555258	-0.12909217	0.00003216	0.00017960
	-0.02057759	1.02738258	-0.00001034	-0.12898751	18.15432739	-0.00006634	0.01779193
	0.00000475	-0.00001036	0.98175711	0.00003211	-0.00006610	17.89146614	-0.00145511
	-0.00080850	-0.00240033	0.00000051	0.99408985	-0.02240279	0.00000452	-0.00000012
	-0.00238978	0.00288512	-0.00000107	-0.02236521	1.02418029	-0.00000878	0.00213538
	0.00000051	-0.00000106	-0.00200445	0.00000449	-0.00000875	0.98205291	-0.00017244
	0.	0.	0.	0.	0.	0.	1.00000000
							0.82725696
136.00	0.99540947	-0.01296795	0.00000345	13.98177874	-0.06239030	0.00001545	0.00007463
	-0.01295659	1.01566252	-0.00000715	-0.06237113	14.06951606	-0.00003065	0.01143198
	0.00000345	-0.00000715	0.98994660	0.00001538	-0.00003049	13.94921875	-0.00093365
	-0.00055210	-0.00191403	0.00000043	0.99679753	-0.01376444	0.00000259	-0.00000185
	-0.00191020	0.00213914	-0.00000086	-0.01375389	1.01415300	-0.00000487	0.00173239
	0.00000042	-0.00000086	-0.00155332	0.00000257	-0.00000485	0.98916716	-0.00014009
	0.	0.	0.	0.	0.	0.	1.00000000
							0.85505559

TIME	LAMBDA MATRIX						
140.00	0.99804726	-0.00685316	0.00000177	9.99439287	-0.02331395	0.00000510	0.00001346
	-0.00685069	1.00753604	-0.00000341	-0.02330425	10.02424085	-0.00000957	0.00619926
	0.00000176	-0.00000341	0.99444710	0.00000516	-0.00000969	9.98161125	-0.00050447
	-0.00033735	-0.00139964	0.00000028	0.99856213	-0.00712795	0.00000113	-0.00000388
	-0.00139865	0.00145539	-0.00000054	-0.00712603	1.00698395	-0.00000204	0.00129233
	0.00000028	-0.00000054	-0.00110583	0.00000112	-0.00000203	0.99448432	-0.00010443
	0.	0.	0.	0.	0.	0.	1.00000000
							0.88814715
144.00	0.99944186	-0.00254705	0.00000050	5.99905431	-0.00515932	0.00000080	-0.00000723
	-0.00254644	1.00255017	-0.00000091	-0.00515649	6.00502777	-0.00000146	0.00237390
	0.00000050	-0.00000091	0.99801144	0.00000082	-0.00000152	5.99608898	-0.00019202
	-0.00016677	-0.00085855	0.00000013	0.99955516	-0.00260286	0.00000028	-0.00000483
	-0.00085843	0.00083093	-0.00000023	-0.00260267	1.00243063	-0.00000049	0.00081090
	0.00000013	-0.00000023	-0.00066152	0.00000028	-0.00000049	0.99801806	-0.00006533
	0.	0.	0.	0.	0.	0.	1.00000000
							0.92738657
148.00	0.99995466	-0.00029156	0.00000002	2.00003460	-0.00018955	0.00000007	-0.00000261
	-0.00029120	1.00026508	-0.00000004	-0.00019436	2.00024730	0.00000012	0.00028078
	0.00000002	-0.00000004	0.99977995	0.00000006	0.	1.99991274	-0.00002254
	-0.00004319	-0.00029220	0.00000001	0.99995894	-0.00029316	0.00000001	-0.00000283
	-0.00029220	0.00026321	-0.00000003	-0.00029317	1.00026098	-0.00000001	0.00028308
	0.00000001	-0.00000003	-0.00021993	0.00000001	-0.00000001	0.99978013	-0.00002269
	0.	0.	0.	0.	0.	0.	1.00000000
							0.97377091
150.00	0.99999999	-0.00000022	-0.00000000	0.00000870	0.00000934	0.00000016	-0.00000004
	0.00000001	0.99999953	-0.00000000	0.00000653	0.00001162	0.00000006	-0.00000002
	0.00000000	-0.00000000	0.99999999	0.00000018	0.00000012	0.	-0.00000000
	0.00000000	0.00000000	0.00000000	0.99999996	-0.00000000	0.00000000	0.00000000
	0.00000000	-0.00000000	-0.00000000	0.00000016	1.00000012	0.00000000	-0.00000000
	0.00000000	-0.00000000	-0.00000000	0.00000000	0.00000000	0.99999997	-0.00000000
	0.	0.	0.	0.	0.	0.	1.00000000
							0.99999999

CONSTANT SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 1 of 13)

TIME	ACCELERATION X 10000/G	ACCEL MAG X 10000/G	POSITION	VELOCITY	MASS	GAMMA
0.	0.3294663	1.2000000	1.0116800	0.0002000	1.0000000	3.1813003
	1.1436068		0.	0.0169900		
	-0.1536729		0.	0.0004900		
4.0000	0.2996616	1.2122336	1.0102957	-0.0008923	0.9899082	2.9816964
	1.1632260		0.0683635	0.0171794		
	-0.1631520		0.0018965	0.0004576		
8.0000	0.2699839	1.2247193	1.0045439	-0.0019827	0.9798163	2.7783701
	1.1819361		0.1373361	0.0172945		
	-0.1734153		0.0036551	0.0004211		
12.0000	0.2407136	1.2374648	0.9944447	-0.0030648	0.9697245	2.5727966
	1.1997095		0.2066208	0.0173355		
	-0.1845896		0.0052599	0.0003806		
16.0000	0.2121415	1.2504784	0.9800452	-0.0041318	0.9596327	2.3664113
	1.2165322		0.2759221	0.0173031		
	-0.1968291		0.0066947	0.0003361		
20.0000	0.1845605	1.2637686	0.9614199	-0.0051765	0.9495409	2.1605851
	1.2323992		0.3449496	0.0171989		
	-0.2103348		0.0079437	0.0002877		
24.0000	0.1582544	1.2773444	0.9386720	-0.0061919	0.9394490	1.9565968
	1.2473055		0.4134216	0.0170258		
	-0.2253736		0.0089908	0.0002352		
28.0000	0.1334840	1.2912150	0.9119336	-0.0071707	0.9293572	1.7556050
	1.2612306		0.4810687	0.0167872		
	-0.2423128		0.0098201	0.0001787		
32.0000	0.1104681	1.3053901	0.8813643	-0.0081062	0.9192654	1.5586216
	1.2741133		0.5476382	0.0164879		
	-0.2616783		0.0104150	0.0001180		
36.0000	0.0893589	1.3196800	0.8471501	-0.0089922	0.9091736	1.3664902
	1.2858062		0.6128982	0.0161334		
	-0.2842544		0.0107585	0.0000530		

CONSTANT SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 2 of 13)

TIME	ACCELERATION X 10000/G	ACCEL MAG X 10000/G	POSITION	VELOCITY	MASS	GAMMA
40.0000	0.0702083	1.3346951	0.8095002	-0.0098231	0.8990817	1.1798739
	1.2959922		0.6766404	0.0157302		
	-0.3112649		0.0108322	-0.0000169		
43.9836	0.	0.	0.7688183	-0.0105915	0.8890314	0.9999871
	0.		0.7384327	0.0152870		
	-0.		0.0106170	-0.0000921		
47.9836	0.	0.	0.7249945	-0.0113093	0.8890314	0.8351683
	0.		0.7981155	0.0145490		
	-0.		0.0102284	-0.0001021		
51.9836	0.	0.	0.6784327	-0.0119604	0.8890314	0.6733078
	0.		0.8547866	0.0137824		
	-0.		0.0098012	-0.0001114		
55.9836	0.	0.	0.6293996	-0.0125452	0.8890314	0.5150285
	0.		0.9083465	0.0129944		
	-0.		0.0093382	-0.0001199		
59.9836	-0.	0.	0.5781581	-0.0130648	0.8890314	0.3621109
	0.		0.9587220	0.0121913		
	-0.		0.0088425	-0.0001278		
63.9836	-0.	0.	0.5249653	-0.0135211	0.8890314	0.2218942
	0.		1.0058651	0.0113791		
	-0.		0.0083170	-0.0001349		
67.9836	-0.	0.	0.4700703	-0.0139164	0.8890314	0.1360315
	-0.		1.0497496	0.0105629		
	-0.		0.0077645	-0.0001413		
71.9836	-0.	0.	0.4137118	-0.0142533	0.8890314	0.1965763
	-0.		1.0903695	0.0097475		
	-0.		0.0071878	-0.0001470		
75.9836	-0.	0.	0.3561174	-0.0145349	0.8890314	0.3321028
	-0.		1.1277360	0.0089369		
	-0.		0.0065897	-0.0001520		

CONSTANT SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 3 of 13)

TIME	ACCELERATION x 10000/G	ACCEL MAG x 10000/G	POSITION	VELOCITY	MASS	GAMMA
79.9836	-0.	0.	0.2975026	-0.0147641	0.8890314	0.4848716
	-0.		1.1618757	0.0081346		
	-0.		0.0059727	-0.0001564		
83.9836	-0.	0.	0.2380704	-0.0149441	0.8890314	0.6447890
	-0.		1.1928279	0.0073436		
	-0.		0.0053393	-0.0001602		
87.9836	-0.	0.	0.1780108	-0.0150783	0.8890314	0.8099249
	-0.		1.2206427	0.0065663		
	-0.		0.0046918	-0.0001634		
91.9836	-0.	0.	0.1175011	-0.0151697	0.8890314	0.9800643
	-0.		1.2453792	0.0058048		
	0.		0.0040325	-0.0001661		
92.4382	-0.0663608	1.3497836	0.1106031	-0.0151775	0.8890314	0.9997274
	-1.3481456		1.2479987	0.0057193		
	0.0039453		0.0039569	-0.0001664		
96.4382	-0.0668646	1.3652816	0.0497593	-0.0152381	0.8789395	1.1624155
	-1.3633135		1.2688497	0.0047086		
	0.0299901		0.0032921	-0.0001651		
100.4382	-0.0673457	1.3811396	-0.0112534	-0.0152624	0.8688477	1.3267161
	-1.3786084		1.2856873	0.0037126		
	0.0494975		0.0026430	-0.0001588		
104.4382	-0.0674261	1.3973703	-0.0722951	-0.0152530	0.8587559	1.4928752
	-1.3942510		1.2985693	0.0027308		
	0.0645107		0.0020269	-0.0001487		
108.4382	-0.0668236	1.4139870	-0.1332352	-0.0152120	0.8486640	1.6611092
	-1.4103456		1.3075527	0.0017631		
	0.0762839		0.0014573	-0.0001357		
112.4382	-0.0653175	1.4310037	-0.1939510	-0.0151411	0.8385722	1.8315889
	-1.4269450		1.3126926	0.0008090		
	0.0856335		0.0009446	-0.0001202		

CONSTANT SPECIFIC IMPULSE TRAJECTORY DATA (Sheet 4 of 13)

TIME	ACCELERATION X 10000/G	ACCEL MAG X 10000/G	POSITION	VELOCITY	MASS	GAMMA
116.4382	-0.0627299	1.4484350	-0.2543258	-0.0150417	0.8284804	2.0044337
	-1.4440769		1.3140425	-0.0001319		
	0.0931166		0.0004981	-0.0001028		
120.4382	-0.0589137	1.4662961	-0.3142485	-0.0149151	0.8183886	2.1797113
	-1.4617547		1.3116541	-0.0010602		
	0.0991289		0.0001240	-0.0000840		
124.4382	-0.0537447	1.4846033	-0.3736117	-0.0147621	0.8082967	2.3574390
	-1.4799833		1.3055773	-0.0019762		
	0.1039603		-0.0001719	-0.0000638		
128.4382	-0.0471157	1.5033733	-0.4323112	-0.0145834	0.7982049	2.5375865
	-1.4987609		1.2958602	-0.0028804		
	0.1078295		-0.0003852	-0.0000427		
132.4382	-0.0389321	1.5226241	-0.4902453	-0.0143795	0.7881131	2.7200791
	-1.5180805		1.2825497	-0.0037730		
	0.1109052		-0.0005126	-0.0000208		
136.4382	-0.0291087	1.5423743	-0.5473137	-0.0141506	0.7780213	2.9047992
	-1.5379303		1.2656914	-0.0046543		
	0.1133198		-0.0005512	0.0000016		
140.4382	-0.0175671	1.5626435	-0.6034168	-0.0138969	0.7679294	3.0915895
	-1.5582940		1.2453301	-0.0055245		
	0.1151789		-0.0004992	0.0000245		
144.4382	-0.0042331	1.5834526	-0.6584558	-0.0136185	0.7578376	3.2802538
	-1.5791505		1.2215100	-0.0063837		
	0.1165680		-0.0003551	0.0000476		
148.4382	0.0109641	1.6048235	-0.7123314	-0.0133152	0.7477458	3.4705588
	-1.6004745		1.1942753	-0.0072318		
	0.1175568		-0.0001182	0.0000709		
150.0000	0.0174188	1.6133252	-0.7330299	-0.0131900	0.7438054	3.5452521
	-1.6089210		1.1827242	-0.0075599		
	0.1178471		-0.0000004	0.0000800		

INITIAL STATE	FINAL STATE	TARGET STATE	ETA	TARGET XI	INITIAL XI	NU X 10000
1.011679992	-0.733029932	-0.733029999	0.405223221	0.000000067	-0.000010386	0.009879917
0.	1.182724193	1.182719991	0.735272802	0.000004202	-0.000020511	0.010000678
0.	-0.000000387	0.	0.000009537	-0.000000387	0.000000492	-0.000079870
0.000199996	-0.013190001	0.013189994	-0.013189413	-0.000000007	0.000000343	0.259339944
0.016989999	-0.007559866	0.007560000	-0.003007151	0.000000127	0.000000142	0.900211900
0.000489995	0.000079989	0.000079997	-0.000664875	-0.000000007	0.000000000	-0.120964438

J	MSTAR MATRIX					
0.2406927E-06	1204984.781250	630053.359375	28086.504150	-27525.345459	-11522.029175	-43.674442
	630053.359375	770784.320313	32584.504395	-16717.953613	-9471.375488	284.025017
	28086.504150	32584.504395	297408.890625	-860.752678	-208.700434	-14.018351
	-27525.345459	-16717.953613	-860.752678	664.792206	271.209042	-0.573601
	-11522.029175	-9471.375488	-208.700434	271.209042	151.921152	-0.030332
	-43.674442	284.025017	-14.018351	-0.573601	-0.030332	45.227303

TIME	LAMBDA MATRIX						
0.	2.91140488	1.50397712	0.02672973	150.29245567	60.52573776	0.92882475	-0.12526201
	4.92616403	1.73948872	0.04141060	144.99959183	281.57543564	1.94516391	-0.52578054
	0.04297525	0.02107574	-0.76021926	1.20710492	1.19105780	64.44785309	0.03073376
	-0.01278768	0.00330262	0.00001433	-0.05722576	-0.36928067	0.00044231	0.00092898
	0.06791733	0.01312629	0.00057152	1.84578894	3.94320241	0.02896937	-0.00410277
	0.00021904	0.00012413	-0.01286746	0.00712010	0.00914412	-0.22445370	-0.00026251
	0.	0.	0.	0.	0.	0.	1.00000000
							0.74380588
4.00	2.57524660	1.55284175	0.02725744	139.32434273	54.39872313	0.82050359	-0.10304333
	4.58585602	2.04006922	0.04306673	125.96991920	274.00204468	1.77583489	-0.45359371
	0.04007082	0.02224465	-0.68496913	1.04091865	1.10428734	67.33925819	0.02899026
	-0.01267178	0.00287846	0.00001494	-0.00631635	-0.38164469	0.00038380	0.00084076
	0.06356646	0.02237077	0.00059729	1.58273663	3.86203164	0.02662689	-0.00310049
	0.00020339	0.00013351	-0.01309417	0.00627561	0.00862812	-0.17251045	-0.00025381
	0.	0.	0.	0.	0.	0.	1.00000000
							0.75138874
8.00	2.25670415	1.56326279	0.02682918	129.66696930	48.15428305	0.71204004	-0.08329055
	4.23096317	2.29726288	0.04363953	108.33219624	265.31274414	1.60207514	-0.38337488
	0.03690860	0.02301519	-0.60685384	0.88688251	1.01363197	69.92378235	0.02697324
	-0.01258052	0.00244462	0.00001552	0.04418145	-0.39229539	0.00032282	0.00075057
	0.05898702	0.02607547	0.00060897	1.33756210	3.76495603	0.02420983	-0.00211874
	0.00018869	0.00014085	-0.01326062	0.00549166	0.00807873	-0.11978063	-0.00024657
	0.	0.	0.	0.	0.	0.	1.00000000
							0.75912781
12.00	1.95960015	1.53893495	0.02561627	121.24209976	41.93897438	0.60691645	-0.06592420
	3.86631641	2.51025754	0.04321894	92.13522530	255.68289185	1.42804390	-0.31522496
	0.03354514	0.02337336	-0.52635048	0.74591855	0.92071606	72.19090176	0.02466618
	-0.01250504	0.00199412	0.00001583	0.09434896	-0.40117937	0.00025998	0.00065815
	0.05423345	0.02922342	0.00060757	1.11107272	3.65417036	0.02177258	-0.00115881
	0.00017453	0.00014625	-0.01336629	0.00476530	0.00750390	-0.06650651	-0.00024105
	0.	0.	0.	0.	0.	0.	1.00000000
							0.76702796
16.00	1.68730225	1.48405859	0.02379220	113.95706177	35.88355255	0.50792462	-0.05084677
	3.49707559	2.67886341	0.04191270	77.40779018	245.28989601	1.25750521	-0.24923350
	0.03004804	0.02331543	-0.44398800	0.61869988	0.82720044	74.13210487	0.02205136
	-0.01243531	0.00152226	0.00001568	0.14422954	-0.40821978	0.00019676	0.00056329
	0.04936574	0.03180374	0.00059433	0.90384672	3.53192577	0.01936506	-0.00022191
	0.00016056	0.00014973	-0.01341116	0.00409511	0.00691129	-0.01293137	-0.00023752
	0.	0.	0.	0.	0.	0.	1.00000000
							0.77509426

LAMBDA MATRIX

TIME							
20.00	1.44258374	1.40328616	0.02152888	107.70687008	30.10104799	0.41716420	-0.03794355
	3.12360277	2.80363336	0.03984635	64.15758801	234.31046867	1.09375639	-0.18547811
	0.02649362	0.02285067	-0.36034279	0.50561084	0.73473569	75.74109364	0.01910933
	-0.01236057	0.00102684	0.00001501	0.19382481	-0.41332591	0.00013518	0.00046586
	0.04444873	0.03381354	0.00057068	0.71621317	3.40050146	0.01703184	0.00069086
	0.00014655	0.00015128	-0.01239585	0.00348084	0.00630859	0.04070257	-0.00023628
	.	0.	0.	0.	0.	0.	1.00000000
							0.78333203
24.00	1.22749384	1.30162387	0.01899218	102.37684345	24.68512559	0.33605758	-0.02708400
	2.76629755	2.88595015	0.03716154	52.37072802	222.91746521	0.93955950	-0.12402432
	0.02296355	0.02200346	-0.27603060	0.40671846	0.64490566	77.01395607	0.01581815
	-0.01226995	0.00050836	0.00001378	0.24309298	-0.41640365	0.00007743	0.00036582
	0.03954795	0.03525940	0.00053829	0.54823777	3.26216972	0.01481128	0.00157857
	0.00013238	0.00015065	-0.01332160	0.00292291	0.00570363	0.09415685	-0.00023770
	0.	0.	0.	0.	0.	0.	1.00000000
							0.79174679
28.00	1.04325309	1.18429068	0.01633678	97.84574127	19.70895171	0.26538429	-0.01812351
	2.41540232	2.92806634	0.03401272	42.01196861	211.27649689	0.79708305	-0.06492565
	0.01954071	0.02081428	-0.19169535	0.32175921	0.55916464	77.94930553	0.01215204
	-0.01215298	-0.00003000	0.00001206	0.29194928	-0.41736620	0.00002561	0.00026324
	0.03473246	0.03615843	0.00049902	0.39971624	3.11915573	0.01273467	0.00244035
	0.00011806	0.00014841	-0.01319043	0.00242198	0.00510442	0.14719949	-0.00024217
	0.	0.	0.	0.	0.	0.	1.00000000
							0.80034429
32.00	0.89018834	1.05654374	0.01370098	93.98920536	15.22467625	0.20533622	-0.01090578
	2.08079273	2.93308762	0.03056238	33.02576685	199.54242325	0.66785981	-0.00822449
	0.01630449	0.01933955	-0.10799456	0.25014351	0.47877272	78.54836369	0.00807942
	-0.01200037	-0.00058255	0.00000996	0.34026928	-0.41614461	-0.00001853	0.00015827
	0.03306735	0.03653872	0.00045487	0.27017725	2.97359428	0.01082560	0.00327548
	0.00010372	0.00014398	-0.01300514	0.00197843	0.00451896	0.19960815	-0.00025018
	0.	0.	0.	0.	0.	0.	1.00000000
							0.80913056
36.00	0.76771811	0.92348307	0.01120204	90.68339729	11.26366961	0.15559280	-0.00526577
	1.76677044	2.90489489	0.02697484	25.33813953	187.85610390	0.55276671	0.04604603
	0.01332605	0.01764986	-0.02558258	0.19097896	0.40473529	78.81498051	0.00355951
	-0.01180448	-0.00114142	0.00000764	0.38789446	-0.41269724	-0.00005380	0.00005121
	0.02561355	0.03643907	0.00040791	0.15889529	2.82748622	0.00909946	0.00408335
	0.00008959	0.00013768	-0.01276924	0.00159192	0.00395501	0.25117313	-0.00026231
	.	0.	0.	0.	0.	0.	1.00000000
							0.81811189

LAMBDA MATRIX

TIME							
40.00	0.67439517	0.78985372	0.00853250	87.80855846	7.83754575	0.11541282	-0.00103347
	1.47687997	2.84800640	0.02340957	18.85937977	176.34154129	0.45203084	0.09785818
	0.01066400	0.01582710	0.05490752	0.14311247	0.33775154	78.75558758	-0.00146326
	-0.01155999	-0.00169709	0.00000528	0.43464027	-0.40701743	-0.00007965	-0.00005752
	0.02142482	0.03590788	0.00036025	0.06491493	2.68265742	0.00756323	0.00486337
	0.00007597	0.00012972	-0.01248701	0.00126103	0.00341968	0.30170032	-0.00027927
	0.	0.	0.	0.	0.	0.	1.00000000
							0.82729485
43.98	0.60822709	0.66038553	0.00696558	85.26227951	4.95079678	0.08385231	0.00195158
	1.21479134	2.76776245	0.02002773	13.50728762	165.14914513	0.36559144	0.14698502
	0.00836970	0.01396792	0.13256820	0.10532475	0.27840662	78.38125420	-0.00704494
	-0.01126541	-0.00223685	0.00000305	0.48012075	-0.39917556	-0.00009619	-0.00016694
	0.01756071	0.03500582	0.00031410	-0.01262803	2.54129854	0.00622080	0.00561188
	0.00006327	0.00012048	-0.01216470	0.00098404	0.00292095	0.35081415	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
47.98	0.56580919	0.53748178	0.00531537	82.92168331	2.55790892	0.05939718	0.00195158
	0.97991890	2.66868913	0.01691478	9.12743104	154.27093506	0.29180950	0.14698502
	0.00644115	0.01214047	0.20753174	0.07582544	0.22621287	77.69998074	-0.00704494
	-0.01091830	-0.00275477	0.00000107	0.52450503	-0.38918272	-0.00010434	-0.00016694
	0.01402131	0.03378720	0.00027073	-0.07567489	2.40361872	0.00505225	0.00561188
	0.00005170	0.00011032	-0.01180487	0.00075452	0.00245909	0.39876457	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
51.98	0.54407634	0.42420478	0.00396898	80.70822144	0.63806820	0.04092408	0.00195158
	0.77365590	2.55621725	0.01411583	5.62974954	143.81746864	0.22985394	0.14698502
	0.00486106	0.01040118	0.27913975	0.05333067	0.18116477	76.72547531	-0.00704494
	-0.01052217	-0.00323985	-0.00000060	0.56740172	-0.37718119	-0.00010516	-0.00016694
	0.01083826	0.03231935	0.00023086	-0.12527412	2.27133292	0.00405026	0.00561188
	0.00004144	0.00009961	-0.01141369	0.00056867	0.00203907	0.44521134	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
55.98	0.53953078	0.32222193	0.00289152	78.54615879	-0.85082285	0.02728672	0.00195158
	0.59533154	2.43505523	0.01163343	2.90090171	133.83275032	0.17846027	0.14698502
	0.00358973	0.00878114	0.34716061	0.03652515	0.14284395	75.47164631	-0.00704494
	-0.01008113	-0.00368462	-0.00000193	0.60862250	-0.36331768	-0.00009998	-0.00016694
	0.00801430	0.03066332	0.00019466	-0.16286011	2.14531443	0.00320043	0.00561188
	0.00003253	0.00008867	-0.01099586	0.00042117	0.00166245	0.49003857	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729

TIME	LAMBDA MATRIX						
59.98	0.54878215	0.23247616	0.00204702	76.37358570	-1.95605271	0.01748132	0.00195158
	0.44361547	2.30934328	0.00946108	0.83159401	124.34305859	0.13637270	0.14698502
	0.00258631	0.00730146	0.41143232	0.02425554	0.11072805	73.95318890	-0.00704494
	-0.00960050	-0.00408279	-0.00000291	0.64799797	-0.34776641	-0.00009017	-0.00016694
	0.00554296	0.02887467	0.00016220	-0.18985911	2.02620277	0.02487940	0.00561188
	0.00002494	0.00007776	-0.01055592	0.00030664	0.00132962	0.53314880	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
63.98	0.56866411	0.15529413	0.00140010	74.14173794	-2.72740656	0.01064718	0.00195158
	0.31669438	2.18260717	0.00758515	-0.68109927	115.35933590	0.10237615	0.14698502
	0.00181071	0.00597453	0.47185565	0.01553092	0.08422818	72.18531895	-0.00704494
	-0.00708650	-0.00442940	-0.00000355	0.68538199	-0.33072414	-0.00007712	-0.00016694
	0.00341052	0.02700261	0.00013340	-0.20765599	1.91442786	0.00189794	0.00561188
	0.00001863	0.00006712	-0.01009812	0.00021989	0.00103997	0.57446218	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
67.98	0.59631532	0.09050305	0.00091723	71.81392765	-3.21494147	0.00606191	0.00195158
	0.21243273	2.05775318	0.00598679	-1.73218359	106.87966251	0.07532129	0.14698502
	0.00122499	0.00480525	0.52838653	0.00951675	0.06272136	70.18353367	-0.00704494
	-0.00854589	-0.00472086	-0.00000390	0.72065447	-0.31240474	-0.00006210	-0.00016694
	0.00159794	0.02508960	0.00010816	-0.21756992	1.81023610	0.00141595	0.00561188
	0.00001349	0.00005694	-0.00962640	0.00015600	0.00079200	0.61391526	-0.00030193
	.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
71.98	0.62322496	0.03754457	0.00056762	69.36421108	-3.46721703	0.00313183	0.00195158
	0.12351356	1.93709265	0.00464370	-2.40770552	98.89169025	0.05414137	0.14698502
	0.00079428	0.00379256	0.58102818	0.00552438	0.04557720	67.96341228	-0.00704494
	-0.00798568	-0.00495488	-0.00000396	0.75372300	-0.29303384	-0.00004628	-0.00016694
	0.00008254	0.02317129	0.00008629	-0.22083599	1.71371779	0.00102812	0.00561188
	0.00000942	0.00004740	-0.00914439	0.00011048	0.00058354	0.65145975	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
75.98	0.66525061	-0.00441937	0.00032376	66.77597809	-3.52996457	0.00138009	0.00195158
	0.06255224	1.82238841	0.00353150	-2.78425306	91.37495422	0.03786372	0.14698502
	0.00048742	0.00293075	0.62982340	0.00299754	0.03217949	65.54043674	-0.00704494
	-0.00741285	-0.00513043	-0.00000380	0.78452314	-0.27284366	-0.00003067	-0.00016694
	-0.00116049	0.02127687	0.00006757	-0.21859363	1.62483360	0.00072140	0.00561188
	0.00000631	0.00003859	-0.00865532	0.00007929	0.00041180	0.68706100	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729

TIME	LAMBDA MATRIX						
79.98	0.70261497	-0.03641342	0.00016168	64.04043579	-3.44515914	0.00043279	0.00195158
	0.01218833	1.71491370	0.00262503	-2.92891270	84.30292416	0.02561491	0.14698502
	0.00027718	0.00221066	0.67484725	0.00149636	0.02194214	62.92985201	-0.00704494
	-0.00683414	-0.00524753	-0.00000345	0.81301801	-0.25206836	-0.00001610	-0.00016694
	-0.00215698	0.01942952	0.00005174	-0.21188088	1.54343978	0.00048368	0.00561188
	0.00000401	0.00003063	-0.00816210	0.00005889	0.00027362	0.72069680	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
83.98	0.73988738	-0.05957070	0.00006092	61.15518951	-3.25044063	0.00000495	0.00195158
	-0.02484624	1.61551847	0.00185523	-2.85952311	77.64485073	0.01662252	0.14698502
	0.00014031	0.00162086	0.71620025	0.00068235	0.01432040	60.14655209	-0.00704494
	-0.00625592	-0.00530721	-0.00000297	0.83919699	-0.23093996	-0.00000322	-0.00016694
	-0.00293273	0.01764707	0.00003857	-0.20163216	1.46931063	0.00030387	0.00561188
	0.00000240	0.00002356	-0.00766732	0.00004627	0.00016553	0.75235578	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
87.98	0.77595456	-0.07506856	0.00000443	58.12294960	-2.97880751	-0.00011349	0.00195158
	-0.05067990	1.52469464	0.00132991	-2.74508148	71.36732674	0.01021251	0.14698502
	0.00005734	0.00114849	0.75400240	0.00030208	0.00881862	57.20498371	-0.00704494
	-0.00568406	-0.00531124	-0.00000239	0.86307395	-0.20968486	0.00000753	-0.00016694
	-0.00351280	0.01594263	0.00002778	-0.18867990	1.40215892	0.00017189	0.00561188
	0.00000135	0.00001741	-0.00717320	0.00003893	0.00008388	0.78203628	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
91.98	0.80998588	-0.08407960	-0.00002163	54.95027447	-2.65854064	-0.00007092	0.00195158
	-0.06727093	1.44264095	0.00089422	-2.50640163	65.43558216	0.00580511	0.14698502
	0.00001236	0.00078002	0.78838794	0.00017297	0.00499392	54.11908627	-0.00704494
	-0.00512384	-0.00526203	-0.00000178	0.88468508	-0.18852103	0.00001589	-0.00016694
	-0.00392104	0.01432531	0.00001913	-0.17375883	1.34165314	0.00007872	0.00561188
	0.00000074	0.00001220	-0.00668169	0.00003488	0.00002495	0.80974491	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729
92.44	0.81369649	-0.08474425	-0.00002318	54.58119965	-2.62016737	-0.00006076	0.00195158
	-0.06865256	1.43387088	0.00085220	-2.47550008	64.78174496	0.00540826	0.14698502
	0.00000906	0.00074411	0.79208604	0.00016813	0.00464753	53.75983572	-0.00704494
	-0.00506112	-0.00525320	-0.00000171	0.88700014	-0.18613085	0.00001669	-0.00016694
	-0.00395758	0.01414729	0.00001827	-0.17196800	1.33518130	0.00007021	0.00561188
	0.00000069	0.00001166	-0.00662607	0.00003456	0.00001953	0.81276979	-0.00030193
	0.	0.	0.	0.	0.	0.	1.00000000
							0.83664729

TIME	LAMBDA MATRIX						
96.44	0.844794[illegible]8	-0.08786026	-0.00002822	51.26326418	-2.27339041	0.00004662	0.00190523
	-0.07703024	1.36150396	0.00054020	-2.18198520	59.19383907	0.00265645	0.12821247
	-0.00000846	0.00047509	0.82284898	0.00017315	0.00223625	50.52891636	-0.00684137
	-0.00451913	-0.00514808	-0.00000110	0.90615456	-0.16531209	0.00002230	-0.00012699
	-0.00420068	0.01263297	0.00001170	-0.15560580	1.28165178	0.00001083	0.00521731
	0.00000044	0.00000747	-0.00613919	0.00003237	-0.00001845	0.83829882	-0.00029866
	0.	0.	0.	0.	0.	0.	1.00000000
							0.84625350
100.44	0.87293179	-0.08679160	-0.00002307	47.82679081	-1.92285737	0.00015152	0.00180308
	-0.07768292	1.29746331	0.00021751	-1.86689197	53.87860346	0.00096759	0.11069721
	-0.00001159	0.00028005	0.85057397	0.00021672	0.00074811	47.18109035	-0.00639807
	-0.00399668	-0.00499580	-0.00000053	0.92317924	-0.14500903	0.00002554	-0.00009274
	-0.00431705	0.01121114	0.00000674	-0.13853130	1.23399404	-0.00002558	0.00482913
	0.0000004[illegible]	0.00000415	-0.00565691	0.00003074	-0.00004143	0.86188949	-0.00029090
	0.	0.	0.	0.	0.	0.	1.00000000
							0.85608286
104.44	0.89796612	-0.08247119	-0.00001354	44.28395796	-1.58338903	0.00022548	0.00164765
	-0.07794186	1.24128528	0.00016522	-1.55036210	48.80364704	0.00002261	0.09442192
	-0.00000729	0.00014485	0.87545969	0.00025603	-0.00008395	43.72810650	-0.00580193
	-0.00349614	-0.00479893	-0.00000002	0.93815722	-0.12540510	0.00002663	-0.00006420
	-0.00432426	0.00987957	0.00000311	-0.12121485	1.19184217	-0.00004487	0.00444597
	0.00000050	0.00000162	-0.00517925	0.00002896	-0.00005272	0.88356028	-0.00027941
	0.	0.	0.	0.	0.	0.	1.00000000
							0.86614326
108.44	0.91986263	-0.07572345	-0.00000348	40.64725828	-1.26632422	0.00025926	0.00144832
	-0.07293908	1.19247052	0.00006687	-1.24768470	43.93849230	-0.00042596	0.07937556
	-0.00000037	0.00005655	0.89768386	0.00027174	-0.00047319	40.18095779	-0.00511543
	-0.00301967	-0.00455981	0.00000038	0.95118047	-0.10667387	0.00002588	-0.00004123
	-0.00423734	0.00863519	0.00000059	-0.10406245	1.15484098	-0.00005196	0.00406640
	0.00000065	-0.00000019	-0.00470610	0.00002667	-0.00005535	0.90332950	-0.00026475
	0.	0.	0.	0.	0.	0.	1.00000000
							0.87644290
112.44	0.93860914	-0.06727620	0.00000484	36.92917395	-0.97986957	0.00025565	0.00121882
	-0.06563765	1.15050581	0.00000849	-0.96991049	39.25473547	-0.00056536	0.06555498
	0.00000621	0.00000372	0.91740418	0.00025961	-0.00058363	36.54997063	-0.00438543
	-0.00256926	-0.00428051	0.00000069	0.96234931	-0.08898015	0.00002367	-0.00002357
	-0.00406902	0.00747441	-0.00000102	-0.08742448	1.12264910	-0.00005084	0.00368900
	0.00000080	-0.00000140	-0.00423731	0.00002373	-0.00005195	0.92121492	-0.00024736
	.	0.	0.	0.	0.	0.	1.00000000
							0.88699047

TIME	LAMBDA MATRIX						
116.44	0.95449634	-0.05777188	0.00001036	33.14187384	-0.72952189	0.00022426	0.00097525
	-0.05685833	1.11487755	-0.00002152	-0.72454993	34.72600031	-0.00053161	0.05296548
	0.00001084	-0.00002345	0.93475948	0.00022466	-0.00053710	32.84487486	-0.00364813
	-0.00214686	-0.00396290	0.00000090	0.97177192	-0.07248084	0.00002043	-0.00001082
	-0.00383013	0.00639345	-0.00000194	-0.07160433	1.09493923	-0.00004468	0.00331238
	0.00000093	-0.00000209	-0.00377265	0.00002022	-0.00004481	0.93723350	-0.00022758
	0.	0.	0.	0.	0.	0.	1.00000000
							0.89779499
120.44	0.96750406	-0.04777793	0.00001284	29.29696560	-0.51834778	0.00017682	0.00073434
	-0.04730247	1.08508009	-0.00003255	-0.51607300	30.32793689	-0.00041838	0.04162114
	0.00001293	-0.00003316	0.94987085	0.00017624	-0.00041927	29.07488322	-0.00293217
	-0.00175443	-0.00360869	0.00000100	0.97956419	-0.05732569	0.00001658	-0.00000250
	-0.00352982	0.00538853	-0.00000232	-0.05686530	1.07139991	-0.00003599	0.00293516
	0.00000099	-0.00000235	-0.00331185	0.00001635	-0.00003579	0.95140125	-0.00020572
	0.	0.	0.	0.	0.	0.	1.00000000
							0.90886599
124.44	0.97789086	-0.03779811	0.00001268	25.40533471	-0.34728076	0.00012492	0.00051184
	-0.03757170	1.06062189	-0.00003197	-0.34635556	26.03822637	-0.00028669	0.03154498
	0.00001265	-0.00003207	0.96284315	0.00012428	-0.00028618	25.24875665	-0.00226057
	-0.00139377	-0.00321940	0.00000100	0.98584997	-0.04365807	0.00001255	0.00000202
	-0.00317580	0.00445593	-0.00000230	-0.04343738	1.05173452	-0.00002661	0.00255598
	0.00000098	-0.00000229	-0.00285465	0.00001238	-0.00002640	0.96373311	-0.00018202
	0.	0.	0.	0.	0.	0.	1.00000000
							0.92021342
128.44	0.98588765	-0.02827987	0.00001062	21.47702289	-0.21534445	0.00007781	0.00032134
	-0.02818421	1.04102509	-0.00002540	-0.21504929	21.83647728	-0.00017057	0.02276888
	0.00001057	-0.00002537	0.97376599	0.00007734	-0.00017011	21.37486744	-0.00165204
	-0.00106759	-0.00279649	0.00000090	0.99076138	-0.03161529	0.00000872	0.00000346
	-0.00277463	0.00359218	-0.00000202	-0.03152178	1.03566055	-0.00001789	0.00217352
	0.00000089	-0.00000200	-0.00240074	0.00000861	-0.00001774	0.97424281	-0.00015670
	0.	0.	0.	0.	0.	0.	1.00000000
							0.93184780
132.44	0.99175318	-0.01962265	0.00000757	17.52107239	-0.11989109	0.00004126	0.00017269
	-0.01958831	1.02582730	-0.00001651	-0.11984489	17.70416546	-0.00008585	0.01533371
	0.00000754	-0.00001688	0.98271512	0.00004095	-0.00008541	17.46125746	-0.00112198
	-0.00077758	-0.00234139	0.00000074	0.99443924	-0.02132901	0.00000539	0.00000267
	-0.00233185	0.00279404	-0.00000157	-0.02129567	1.02290946	-0.00001065	0.00178646
	0.00000073	-0.00000156	-0.00194986	0.00000534	-0.00001058	0.98294307	-0.00012996
	0.	0.	0.	0.	0.	0.	1.00000000
							0.94378015

LAMBDA MATRIX

TIME							
136.44	0.99577215	-0.01218261	0.00000442	13.54546130	-0.05675609	0.00001746	0.00007089
	-0.01217319	1.01457843	-0.00000918	-0.05676194	13.62457871	-0.00003411	0.00928519
	0.00000441	-0.00000916	0.98975353	0.00001717	-0.00003406	13.51569176	-0.00068308
	-0.00052634	-0.00185550	0.00000054	0.99703369	-0.01292516	0.00000282	0.00000062
	-0.00185213	0.00205862	-0.00000107	-0.01291624	1.01322444	-0.00000533	0.00139350
	0.00000053	-0.00000107	-0.00150174	0.00000280	-0.00000532	0.98984538	-0.00010199
	0.	0.	0.	0.	0.	0.	1.00000000
							0.95602202
140.44	0.99825418	-0.00627900	0.00000191	9.55694628	-0.02038019	0.00000508	0.00001495
	-0.00627711	1.00683865	-0.00000368	-0.02041575	9.58285654	-0.00000927	0.00469396
	0.00000191	-0.00000367	0.99493254	0.00000481	-0.00000918	9.54570580	-0.00034583
	-0.00031651	-0.00134031	0.00000031	0.99870512	-0.00652400	0.00000111	-0.00000159
	-0.00133949	0.00138330	-0.00000060	-0.00652242	1.00636028	-0.00000201	0.00099334
	0.00000031	-0.00000059	-0.00105613	0.00000111	-0.00000201	0.99496034	-0.00007293
	0.	0.	0.	0.	0.	0.	1.00000000
							0.96858566
144.44	0.99953419	-0.00219446	0.00000045	5.56103855	-0.00409748	0.00000077	-0.00000338
	-0.00219416	1.00217567	-0.00000081	-0.00412033	5.56577826	-0.00000100	0.00161533
	0.00000045	-0.00000081	0.99829292	0.00000060	-0.00000107	5.55865431	-0.00011892
	-0.00015092	-0.00079744	0.00000012	0.99962471	-0.00223955	0.00000025	-0.00000275
	-0.00079734	0.00076585	-0.00000022	-0.00223936	1.00208101	-0.00000042	0.00058471
	0.00000012	-0.00000022	-0.00061285	0.00000025	-0.00000042	0.99829756	-0.00004296
	0.	0.	0.	0.	0.	0.	1.00000000
							0.98148391
148.44	0.99997347	-0.00017813	0.00000001	1.56180631	-0.00006589	0.00000013	-0.00000110
	-0.00017810	1.00016052	-0.00000001	-0.00009413	1.56191456	0.00000006	0.00012957
	0.00000001	-0.00000001	0.99986587	-0.00000004	0.00000015	1.56174278	-0.00000951
	-0.00003264	-0.00022867	0.00000001	0.99997553	-0.00017904	0.00000000	-0.00000148
	-0.00022867	0.00020438	-0.00000001	-0.00017884	1.00015888	-0.00000000	0.00016630
	0.00000001	-0.00000001	-0.00017169	0.00000000	-0.00000001	0.99986595	-0.00001220
	0.	0.	0.	0.	0.	0.	1.00000000
							0.99473032
150.00	0.99999999	-0.00000015	-0.00000000	0.00000594	0.00001007	0.00000001	-0.00000003
	-0.00000000	0.99999993	-0.	0.00000317	0.00001830	0.00000001	-0.00000009
	0.00000000	0.	0.99999999	0.00000009	0.00000003	-0.00000095	0.
	0.00000000	0.00000000	0.00000000	0.99999998	-0.00000000	-0.00000000	0.00000000
	0.00000000	-0.00000000	-0.00000000	0.00000019	1.00000015	0.00000000	-0.00000000
	0.00000000	-0.00000000	-0.00000000	0.00000000	0.00000000	0.99999997	-0.00000000
	0.	0.	0.	0.	0.	0.	1.00000000
							0.99999999

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BIOGRAPHICAL SKETCH

Edgar Dean Mitchell was born in Hereford, Texas on September 17, 1930. He attended high school in Artesia, New Mexico and was graduated from Artesia High School in 1948. He was awarded a scholarship at Carnegie Institute of Technology and received the degree of Bachelor of Science in Industrial Management in 1952.

Edgar Mitchell and Louise Elizabeth Randall were married in December 1951. They now have two daughters, Karlyn and Elizabeth.

Mr. Mitchell entered the United States Navy in September 1952. He was commissioned in May 1953 and received his designation as a naval aviator in July 1954.

Lieutenant Commander Mitchell served as an aviator in Patrol Squadron Twenty Nine and Heavy Attack Squadron Two. In April 1958 he was assigned to Air Development Squadron Five as a research and development pilot for weapons delivery in heavy attack aircraft. He was cited for his work in this assignment.

Lieutenant Commander Mitchell attended the U. S. Naval Postgraduate School as a student and received the degree of Bachelor of Science in Aeronautical Engineering in 1961. He was permitted to pursue further postgraduate work at M. I. T.

Mr. Mitchell was elected to Sigma Gamma Tau and Sigma Xi at M. I. T. and previously to Delta Nu and Kappa Sigma at Carnegie. He is a member of the American Institute of Aeronautics and Astronautics

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